

Note on Ward-Horadam $H(x)$ -binomials' recurrences and related interpretations, II

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Summary

This note is a continuation of [48, 2010]. **Firstly**, we propose $H(x)$ -binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ functions' sequence i.e. any functions' sequence solution of the second order recurrence with functions' coefficients. As a method this comprises $H \equiv H(x=1)$ number sequences V -binomials' recurrence formula determined by the primordial Lucas sequence of the second kind $V = \langle V_n \rangle_{n \geq 0}$ as well as its well elaborated companion fundamental Lucas sequence of the first kind $U = \langle U_n \rangle_{n \geq 0}$ which gives rise in its turn to the known U -binomials' recurrence as in [1, 1878], [6, 1949], [8, 1964], [12, 1969], [17, 1989] or in [18, 1989] etc.

Then we deliver a new type $H(x)$ -binomials' "mixed" recurrence, straightforward formula for special case of Ward-Horadam sequences $H(x) = \langle p^n(x) + q^n(x) \rangle_{n \geq 0}$.

Here general $H(x)$ -binomials' array is appointed by Ward-Horadam sequence of functions which in predominantly considered cases where chosen to be polynomials.

For the sake of combinatorial interpretations and in number theory $H(x=1) = \langle H_n(x=1) \rangle_{n \geq 0}$ is usually considered to be natural or integer numbers valued sequence. Number sequences $H = H(x=1) = \langle H_n \rangle_{n \geq 0}$ were also recently called by several authors: Horadam sequences.

Secondly, we supply a review of selected related combinatorial interpretations of generalized binomial coefficients. We then propose also a kind of transfer of interpretation of p, q -binomial coefficients onto q -binomial interpretations thus bringing us back to György Pólya and Donald Ervin Knuth relevant investigation decades ago.

The list of references is prevailingly indicative (see references therein) and is far from being complete.

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1 General Introduction

1.1. p, q people are followers of Lucas people. There are many authors who use in their investigation the fundamental Lucas sequence $U \equiv \langle n_{p,q} \rangle_{n \geq 0}$ - frequently with different notations - where $n_{p,q} = \sum_{j=0}^{n-1} p^{n-j-1} q^j = U_n$; see Definition 1 and then definitions that follow it. In regard to this a brief intimation is on the way.

Up to our knowledge it was François Édouard Anatole Lucas in [1, 1878] who was the first who had *not only* defined *fibonomial* coefficients as stated in [18, 1989] by Donald Ervin Knuth and Herbert Saul Wilf but who was the first who had defined $U_n \equiv n_{p,q}$ -binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ and had derived a recurrence for them: see page 27, formula (58) [1, 1878]. Then - referring to Lucas - the investigation relative to divisibility properties of relevant number Lucas sequences D, S as well as numbers' D - binomials and numbers' D - multinomials was continued in [89, 1913] by Robert Daniel Carmichel; see pp. 30, 35 and 40 in [89, 1913] for $U \equiv D = \langle D_n \rangle_{n \geq 0}$ and $\binom{n}{k_1, k_2, \dots, k_s}_D$ - respectively. Note there also formulas (10), (11) and (13) which might perhaps serve to derive explicit untangled form of recurrence for the *V-binomial coefficients* $\binom{n}{k}_V \equiv \binom{n}{k}_S$ denoted by primordial Lucas sequence $\langle S_n \rangle_{n \geq 0} = S \equiv V$. Let $F(x=1) \equiv A$ denotes a number sequence as in [3, 1915] by Fontené Georges, [12, 1969] by Henri W. Gould then followed by for example [39, 2005], [40, 2005], [41, 2006] by Jaroslav Seibert and Pavel Trojovský and [42, 2007] by Pavel Trojovský.

A - multinomial coefficients' *recurrences* are not present in that early and other works and up to our knowledge a special case of such appeared at first in [67, 1979] by Anthony G. Shannon. More on that - in what follows after Definition 3.

Significant peculiarity of Lucas originated sequences includes their importance for number theory (see middle-century paper [101] by John H. Halton and recent, this century papers [125, 2010] by Chris Smith and [126, 2010] by Kálmán Györy with Chris Smith and the reader may enjoy also the PhD Thesis [137, 1999] by Anne-Marie Decaillot-Laulagnet). This Lucas originated investigation amalgamates diverse areas of mathematics due to hyperbolic - trigonometric character of these Fonctions Numériques Simplement Périodiques i.e. fundamental and primordial Lucas sequences - as beheld in [1, 1878]. One may track then a piece of further constructions for example in [28, 1999]).

There in [28, 1999] tail formulas (3.12) and (3.14) are illustrating the proved and exploited by Édouard Lucas complete analogy of the V_n and U_n symmetric functions of roots with the circular and hyperbolic functions of order 2. This is so due to Lucas formulas (5) in [1] rewritten in terms of cosh and sinh functions as formulas (3.13) and (3.14) in [28] where there these result from de Moivre one parameter group introduced in [28] in order to pack compactly the addition

formulas (1.6), (1.7) in [28] into abelian group "parcel" encompassing Pafnuty Lvovich Chebyshev polynomials of both kinds.

In this connection see the Section 2 in the recent Ward-Horadam people paper [86, 2009] by Tian-Xiao He, Peter Jau-Shyong Shiue. There in Proposition 2.7. illustrative Example 2.8. with Pafnuty Lvovich Chebyshev polynomials of the first kind the well known recurrence formula (2.28) is equivalent to abelian one-parameter de Moivre matrix group multiplication rule from which the corresponding recurrence (1.7) in [28, 1999] follows.

1.2. We deliver here - continuing the note [48] - a new $H(x)$ -binomials' recurrence formula appointed by Ward-Horadam $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ field of zero characteristic nonzero valued functions' sequence which comprises for $H \equiv H(x=1)$ number sequences case - the V -binomials' recurrence formula determined by the primordial Lucas sequence of the second kind $V = \langle V_n \rangle_{n \geq 0}$ [48, 2010] as well as its well elaborated companion fundamental Lucas sequence of the first kind $U = \langle U_n \rangle_{n \geq 0}$ which gives rise in its turn to the U -binomials' recurrence as in [1, 1878], [6, 1949], [8, 1964], [12, 1969], [17, 1989] or in [18, 1989] and so on.

We do it by following recent applicable work [2, 2009] by Nicolas A. Loehr and Carla D. Savage thought one may - for that purpose - envisage now easy extensions of particular p, q - cases considered earlier - as *for example* the following: the relevant recursions in [6, 1949], in [17, 1989], in [21, 1992] - (recursions (40) and (51)) , or [120, 2000] by John M. Holte (Lemmas 1,2 dealing with U -binomials provide a motivated example for observation Theorem 17 in [2]) One is invited also to track Lemma 1 in [121, 2001] by Hong Hu and Zhi-Wei Sun ; see also corresponding recurrences for p, q -binomials $\equiv U$ -binomials in [1, 1878] or in [47, 2008] **v[1]** by Maciej Dziemiańczuk (compare there (1) and (2) formulas), or see Theorem 1 in [44, 2008] by Roberto Bagsarsa Corcino as well as track the proof of the Corollary 3. in [47, 2009] **v[2]** by Maciej Dziemiańczuk.

This looked for here new $H(x)$ -binomials' overall recurrence formulas (recall: encompassing V -binomials for primordial Lucas sequence V) is not present neither in [1] nor in [2], nor in [3, 1915], nor in [5, 1936], nor in [6, 1949]. Neither we find it in - quoted here contractually by a nickname as "Lucas (p, q)-people" - references [1-47]. Neither it is present - up to this note author knowledge - in all other quoted here contractually by a nickname as "Ward-Horadam -people" - references [53-84]. As for "Lucas (p, q)-people" and "Ward-Horadam -people" references - (including these [n] with $n > 86$ - the distinction which People are which is quite contractual. The nicknames are nevertheless indicatively helpful. We shall be more precise soon - right with definitions are being started.

Interrogation. Might it be so that such an overall recurrence formula does not exist? The method to obtain formulas is more general than actually produced family of formulas? See Theorems 2a and 2b, and Examples 4,2, ... in what follows.

Meanwhile $H(x)$ -binomials' overall recurrence formula for the Ward-Horadam's function sequence $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ follows straightforwardly from the easily proved important observation - the Theorem 17 in [2, 2009] as already had it been remarked in [48, 2010] for the $H \equiv H(x = 1)$ case. Finally, see this note observation named Theorem 2b.

This paper formulas may and should be confronted with Fontené obvious recurrence for complex valued A -binomials $\binom{n}{k}_A$, $A \equiv A(x = 1)$ in [3, 1915] i.e. with (6) or (7) identities in [12, 1969] by Henri W. Gould or with recurrence in [30, 1999] by Alexandru Ioan Lupas, which particularly also stem easily just from the definition of **any** $F(x)$ -binomial coefficients arrays with $F(x) = \langle F_n(x) \rangle_{n \geq 0}$ staying for any field of characteristic zero nonzero valued functions' sequence; $F_n(x) \neq 0$, $n \geq 0$. For $F = F(x = 1)$ -multinomial coefficients automatic definition see [89, 1913] by Robert Daniel Carmichel or then [12, 1969] by Henri W. Gold and finally see [67, 1979] by Anthony G. Shannon, where recurrence is proved for $\binom{n}{k_1, k_2, \dots, k_s}_U$ with U -Lucas fundamental being here complex valued number sequence. For $F(x)$ - multinomial coefficients see [49, 2004] and compare with $F(x)$ -binomials from [30, 1999] or those from [50, 2001].

To this end we supply now two informations (1.3) and (1.4) pertinent ad references and ad nomenclature.

1.3. Ad the number theory and divisibility properties references. For the sake of combinatorial interpretations of F - number sequences as well as their correspondent F -multinomial coefficients and also for the sake of the number theoretic studies of Charles Hermite [87] and with Thomas Jan Stieltjes in [88] or by Robert Daniel Carmichel [89, 1913] or [90, 1919] or that of Morgan Ward [94, 1936], [95, 1939], [96, 1937], [97, 1937], [53, 1954], [98, 1955], [99, 1959] and that of Derric Henry Lehmer [4, 1930], [91, 1933], [92, 1935] or this of Andrzej Bobola Maria Schinzel [103, 1974] and Others' studies *on divisibility properties* - these are the sub-cases $F_n \in \mathbb{N}$ or $F_n \in \mathbb{Z}$ which are being regularly considered at the purpose.

As for the "Others" - see *for example*: [100, 1959], [102, 1973], [104, 1974], [105, 1974], [106, 1974], [107, 1973], [66, 1977], [108, 1977], [69, 1979], [109, 1979], [110, 1980], [70, 1980], [18, 1989], [111, 1991], [115, 1992], [112, 1995], [113, 1999], [114], [116, 1995], [117, 1995], [118, 1995], [119, 1998], [121, 2001], [123, 2006], [124, 2009].

1.4. Ad the name: Ward-Horadam sequence. According to the authors of [86, 2009] it was Mansour [83] who called the sequence $H = \langle a_n \rangle_{n \geq 0}$ defined by (1) a **Horadam's sequence**, as - accordingly to the author of [83] - the number sequence H was introduced in 1965 by Horadam [56] (for special case of Ward-Horadam number sequences see Section 2 in [64, 1974] and see also [84, 2009]), this however notwithstanding the ingress of complex numbers valued F -binomials and F -multinomials into Morgan Ward's systematic *Calculus of sequences* in [5, 1936] and then in 1954 Ward's introduction of "'nomen omen'"

$W \equiv H$ in [53, 1954] integer valued sequences.

Perceive then the appraisal of adequate Morgan Wards' work in the domain by Henri W. Gould [12, 1959] and by Alwyn F. Horadam and Anthony G. Shannon in [65, 1976] or Derrick Henry Lehmer in [93, 1993]. On this occasion note also the **Ward-Horadam** number sequences in [54, 1965] and [57, 1965].

The sequence $H = \langle a_n \rangle_{n \geq 0}$ defined by (2) was called **Horadam's sequence** in [63, 1974] by Anthony J. W. Hilton and in generalization of [63, 1974] which is the work [68, 1979] by Anthony G. Shannon where consequences of the partition of the set of nontrivial solutions of (1) into solutions of Lucas type L (primordial Lucas sequences) and solutions of Fibonacci type F (fundamental Lucas sequences) were studied by both authors. To be more precise: the article [63, 1974] opens the way to consider F -binomial and L -binomial coefficients (not considered in [63, 1974]) while the presentation [68, 1979] opens the way to consider $V_s^{(r)}$ - r -nomial coefficients where $s = 1, \dots, r$ (neither considered in [68, 1979]). It should be though forthwith noted that in another paper [67, 1979] Anthony G. Shannon derives the recursion for fundamental $V_r^{(r)}$ - r -nomial coefficients denoted there as $u^{(r)}$ -multinomial coefficients, where there specifically for $r = 2$ the number sequence $u^{(2)}$ coincides with the fundamental Lucas sequence (i.e. the Lucas sequence of the first kind \equiv of the F -type in the nomenclature of [63, 1974] by Anthony J. W. Hilton). See more - what follows right after this note Definition 3 and finally compare with the way indicated by this note Theorem 2b.

1.5. The sequence $H = \langle a_n \rangle_{n \geq 0}$ defined by the recurrence (2) - below - with the initial conditions $H_0 = 1$ and $H_1 = s$ was exploited in [7, 1962] by Leonard Carlitz. Soon, Leonard Carlitz in [10, 1965] had proved the following H -binomial formula for the sequence $H = \langle a_n \rangle_{n \geq 0}$ defined by the recurrence (2) below with $t = 1$ and *distinct* roots p, q :

$$\prod_{j=0}^n (x - p^j \cdot q^{n-j}) = \sum_{r=0}^{n+1} (-1)^{\frac{r(r+1)}{2}} \cdot \binom{n+1}{r}_{U^*} \cdot x^{n+1-r},$$

where

$$\binom{n}{k}_{U^*} \equiv \binom{n}{k}_U$$

and $U_n^* = p^n - q^n$ while $U_n = \frac{p^n - q^n}{p - q}$. Compare with this note Theorem 2a.

2 Preliminaries

Names: The Lucas sequence $V = \langle V_n \rangle_{n \geq 0}$ is called the Lucas sequence of the second kind - see: [66, 1977, Part I], or **primordial** - see [69, 1979].

The Lucas sequence $U = \langle U_n \rangle_{n \geq 0}$ is called the Lucas sequence of the first kind - see: [66, 1977, Part I], or **fundamental** - see p. 38 in [6, 1949] or see [67, 1979] and [69, 1979].

In the sequel we shall deliver the looked for recurrence for H -binomial coefficients $\binom{n}{k}_H$ determined by the Ward-Horadam sequence H - defined below.

In compliance with Edouard Lucas' [1, 1878] and twenty, twenty first century p, q -people's notation we shall at first review here in brief the general second order recurrence; (compare this review with the recent "Ward-Horadam" peoples' paper [86, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue or earlier p, q -papers [33, 2001] by Zhi-Wei Sun, Hong Hu, J.-X. Liu and [121, 2001] by Hong Hu and Zhi-Wei Sun). And with respect to notation: If in [1, 1878] François Édouard Anatole Lucas had been used $a = \mathbf{p}$ and $b = \mathbf{q}$ notation, he would be perhaps at first glance notified and recognized as a Great Grandfather of all the (p, q) - people. Let us start then introducing reconciling and matched denotations and nomenclature.

$$(1) \quad H_{n+2} = P \cdot H_{n+1} - Q \cdot H_n, \quad n \geq 0 \text{ and } H_0 = a, H_1 = b.$$

which is sometimes being written in $\langle P, -Q \rangle \mapsto \langle s, t \rangle$ notation.

$$(2) \quad H_{n+2} = s \cdot H_{n+1} + t \cdot H_n, \quad n \geq 0 \text{ and } H_0 = a, H_1 = b.$$

We exclude the cases when (2) is recurrence of the first order, therefore we assume that the roots p, q of (5) are distinct $p \neq q$ and $\frac{p}{q}$ is not the root of unity. We shall come back to this finally while formulating this note observation named Theorem 2a.

Simultaneously and collaterally we mnemonically pre adjust the starting point to discuss the $F(x)$ polynomials' case via - if entitled - antecedent " \mapsto action": $H \mapsto H(x), s \mapsto s(x), t \mapsto t(x)$, etc.

$$(3) \quad H_{n+2}(x) = s(x) \cdot H_{n+1}(x) + t(x) \cdot H_n, \quad n \geq 0, \quad H_0 = a(x), H_1 = b(x).$$

enabling recovering explicit formulas also for sequences of polynomials correspondingly generated by the above linear recurrence of order 2 - with Tchebysh-eff polynomials and the generalized Gegenbauer-Humbert polynomials included. See for example Proposition 2.7 in the recent Ward-Horadam peoples' paper [86, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue.

The general solution of (1): $H(a, b; P, Q) = \langle H_n \rangle_{n \geq 0}$ is being called throughout this paper - **Ward-Horadam number's sequence**.

The general solution $H(x) \equiv H(a(x), b(x); s(x), t(x)) = \langle H_n(x) \rangle_{n \geq 0}$ of the recurrence (3) is being called throughout this paper - **Ward-Horadam functions' sequence**. It is then to be noted here that ideas germane to *special*

Ward-Horadam polynomials sequences of the [77] paper were already explored in some details in [56]. For more on special Ward-Horadam *polynomials sequences* by Alwyn F. Horadam - consult then: [61], [71, 1985], [72], [78] or see *for example* the following papers and references therein: recent papers [84, 2009] by Tugba Horzum and Emine Gökçen Kocer and [85, 2009] by Gi-Sang Cheon, Hana Kim and Louis W. Shapiro. For *Ward-Horadam functions sequences* [86, 2009] by Tiang-Xiao He and Peter Jau-Shyong Shiue who however there then concentrate on on *special Ward-Horadam polynomials sequences* only. Other "polynomial" references shall appear in the course of further presentation.

For example - in [136, 2010] Johann Cigler considers special Ward-Horadam *polynomials sequences* and among others he supplies the tiling combinatorial interpretation of these special Ward-Horadam *polynomials sequences* which are q -analogues of the Fibonacci and Lucas polynomials introduced in [134, 2002] and [135, 2003] by Johann Cigler.

In the paper [81, 2003] Johann Cigler introduces "abstract Fibonacci polynomials" - interpreted in terms of Morse coding sequences monoid with concatenation (monominos and dominos tiling then) Cigler's abstract Fibonacci polynomial sare monoid algebra over reals valued polynomials with straightforward Morse sequences i.e. tiling recurrence originated (1.6) "addition formula"

$$F_{m+n}(a, b) = F_{m+1}(a, b) \cdot F_m(a, b) + b \cdot F_{n-1}(a, b) \cdot F_n(a, b),$$

which is attractive and seductive to deal with within the context of this paper Theorem 1 below.

From the characteristic equation of (1)

$$(4) \quad x^2 = P \cdot x - Q,$$

written by some of p, q -people as

$$(5) \quad x^2 = s \cdot x + t$$

we readily find the Binet form solution of (2) (see (6) in [84, 2009]) which is given by (6) and (7).

$$(6) \quad H_n(a, b; P, Q) \equiv H_n(A, B; p, q) = Ap^n + Bq^n, \quad n \geq 0, H_0 = a, H_1 = b.$$

where p, q are roots of (5) and we have assumed since now on that $p \neq q$ and $\frac{p}{q}$ is not the root of unity (see Lemma for example lemma 1 in [70, 1980] by Fritz Beukers). Hence for established $p \neq q$ we shall use sometimes the shortcuts

$$H(A, B) \equiv H_n(A, B; p, q) = (p - q) \cdot U_n(A, -B; p, q) \equiv (p - q) \cdot U_n(A, -B).$$

As for the case $p = q$ included see for example Proposition 2.1 in [86, 2009] and see references therein.

Naturally : $p + q = P \equiv s$, $p \cdot q = Q \equiv -t$ and

$$(7) \quad A = \frac{b - qa}{p - q} , \quad B = -\frac{b - pa}{p - q}.$$

hence we may and we shall use the following conventional identifications-abbreviations

$$(8) \quad \begin{aligned} H(A, B) &\equiv H(a, b; P, Q) \equiv H(A, B; P, Q) \equiv H(A, B; p, q) = \\ &= (p - q) \cdot U_n(A, -B; p, q) \equiv (p - q) \cdot U_n(A, -B). \end{aligned}$$

It is obvious that the exponential generating function for Ward-Horadam sequence H reads:

$$(9) \quad E_H(A, B; p, q)[x] = A \exp[p \cdot x] + B \exp[q \cdot x].$$

The derivation of the formula for ordinary generating function for Ward-Horadam sequence is a standard task and so we have (compare with (5) in [84, 2009] by Tugba Horzum and Emine Gökçen Kocer or put $r = 1$ in Theorem 1 from [82, 2003] by Stănică)

$$(10) \quad G_H(a, b; P, Q)[x] = \frac{a + (b - aP)x}{1 - P \cdot x + Q \cdot x^2} = \frac{a + (b - a[p + q])x}{1 - P \cdot x + p \cdot q \cdot x^2}.$$

where from we decide an identification-abbreviation

$$G_H[x] \equiv G_H(A, B; p, q)[x] \equiv G_H(a, b; P, Q)[x].$$

Naturally - in general $H(A, B; p, q) \neq H(A, B; q, p)$.

If $H(A, B; p, q) = H(A, B; q, p)$ we then call the Ward-Horadam sequence p, q -symmetric and thus we arrive to Lucas ($A = B = 1$) *Théorie des Fonctions Numériques Simplement Périodiques* [1, 1878].

In [1, 1878] Edouard Lucas considers Lucas sequence of the second kind $V = \langle V_n \rangle_{n \geq 0}$ (second kind - see: [66, 1977, Part I]) as well as its till now well elaborated companion Lucas sequence of the first kind $U = \langle U_n \rangle_{n \geq 0}$ (first kind - see: [66, 1977, Part I]) which gives rise in its turn to the U -binomials' recurrence (58) in [1, 1878].

See then [6, 1949], [8, 1964], [12, 1969], [17, 1989] or in [18, 1989] etc. For example - for the relations between U_{n+1} and V_n number sequences and for

the explicit form of ordinary generating functions of their powers see [14, 1977] by Blagoj S. Popow, where the characteristic equation for both H_n recurrent number sequences of second order is of general form $a \cdot z^2 + b \cdot z + c = 0$.

The p, q -symmetric sequences from [1, 1878] i.e the Lucas sequence of the second kind ($A = B = 1$)

$$(11) \quad H_n(2, P; p, q) = V_n = p^n + q^n.$$

and the Lucas sequence of the first kind ($A = -B = 1$)

$$(12) \quad H_n(0, 1; p, q) = U_n = \frac{p^n - q^n}{p - q},$$

where called by Lucas [1, 1878] the *simply periodic numerical functions* because of

[quote] *at the start, the complete analogy of these symmetric functions with the circular and hyperbolic functions.* [end of quote].

More ad Notation 1. The letters a, b $a \neq b$ in [1, 1878] denote the roots of the equation $x^2 = Px - Q$ then $(a, b) \mapsto (u, v)$ in [2, 2009] and u, v stay there for the roots of the equation $x^2 = \ell x - 1$.

We shall use here the identification $(a, b) \equiv (p, q)$ i.e. p, q denote the roots of $x^2 = Px - Q$ as is common in "Lucas (p, q) -people" publications.

For Lucas (p, q) -people then the following U -identifications are expediency natural:

Definition 1

$$(13) \quad n_{p,q} = \sum_{j=0}^{n-1} p^{n-j-1} q^j = U_n = \frac{p^n - q^n}{p - q}, \quad 0_{p,q} = U_0 = 0, \quad 1_{p,q} = U_1 = 1,$$

where p, q denote now the roots of the equation $x^2 = P \cdot x - Q \equiv x^2 = sx + t$ hence $p + q = s \equiv P$, $pq = Q \equiv -t$ and the empty sum convention was used for $0_{p,q} = 0$. Usually one assumes $p \neq q$. In general also $s \neq t$ - though according to the context [17, 1989] $s = t$ may happen to be the case of interest.

The Lucas U -binomial coefficients $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$ are then defined as follows: ([1, 1878], [3, 1915], [5, 1936], [6, 1949], [8, 1964], [12, 1969] etc.)

Definition 2 Let U be as in [1, 1878] i.e $U_n \equiv n_{p,q}$ then U -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(14) \quad \binom{n}{k}_U \equiv \binom{n}{k}_{p,q} = \frac{n_{p,q}!}{k_{p,q}! \cdot (n-k)_{p,q}!} = \frac{n_{p,q}^k}{k_{p,q}!}$$

where $n_{p,q}! = n_{p,q} \cdot (n-1)_{p,q} \cdot \dots \cdot 1_{p,q}$ and $n_{p,q}^k = n_{p,q} \cdot (n-1)_{p,q} \cdot \dots \cdot (n-k+1)_{p,q}$ and $\binom{n}{k}_U = 0$ for $k > n$.

Definition 3 Let V be as in [1, 1878] i.e $V_n = p^n + q^n$, hence $V_0 = 2$ and $V_1 = p+q = s$. Then V -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(15) \quad \binom{n}{k}_V = \frac{V_n!}{V_k! \cdot V_{(n-k)}!} = \frac{V_n^k}{V_k!}$$

where $V_n! = V_n \cdot V_{n-1} \cdot \dots \cdot V_1$ and $V_n^k = V_n \cdot V_{n-1} \cdot \dots \cdot V_{n-k+1}$ and $\binom{n}{k}_V = 0$ for $k > n$.

One automatically generalizes number F -binomial coefficients' array to functions $F(x)$ -**multinomial** coefficients' array (see [49, 2004] and references to umbral calculus therein) while for *number sequences* $F = F(x = 1)$ the F -multinomial coefficients see p. 40 in [89, 1913] by Robert Daniel Carmichel, see [5, 1936] by Morgan Ward and [12, 1969] by Henri W. Gould.

In [67, 1979]) Anthony G. Shannon considers the special case of number sequences, where the $F_n = H_{s,n}^{(r)}$, $n = 0, 1, 2, \dots$, $1 \leq s \leq r$, which are constituting r basic number sequences - satisfy a linear homogeneous recurrence relation of the order r :

$$H_{s,n}^{(r)} = \sum_{j=1}^r (-1)^{j+1} P_{r,j} \cdot H_{s,n-j}^{(r)}, \quad n > r,$$

with initial conditions $H_{s,n}^{(r)} = \delta_{s,n}$ for $1 \leq n \leq r$ and where $P_{r,j}$ are arbitrary integers.

For $r = 2$ the above coincides with this note recurrence (1) where $H_{1,1}^{(2)} = H_1 = 1$, $H_{1,2}^{(2)} = H_0 = 0$ for $s = 1$ and $H_{2,1}^{(2)} = H_0 = 0$, $H_{2,2}^{(2)} = H_1 = 1$ for $s = 2$. Because of that latter Anthony G. Shannon designates the number sequence $\langle H_{r,n}^{(r)} \rangle \equiv \langle u_n^{(r)} \rangle$ as the fundamental sequence by the analogy with Lucas' second-order fundamental sequence $\langle H_{r,n}^{(2)} \equiv U_n \rangle$.

Anthony G. Shannon derives then in [67, 1979] the recursion for $u^{(r)}$ -multinomial coefficients (see Theorem on page 346), where there specifically for $r = 2$ the number sequence $u^{(2)}$ coincides with the fundamental Lucas sequence (i.e. the Lucas sequence of the first kind).

As for other here relevant works see [111, 1991] by Shiro Ando and Daihachiro Sato. Note also x -Fibonomial coefficients array from [50, 2001] by Thomas M. Richardson.

$\psi(x)$ -**multinomial Remark.**

Considerations in [67, 1979] by Anthony G. Shannon and an application in [50, 2001] by Thomas M. Richardson as well as relevance to umbral calculus [182, 2003] or [183, 2001] constitute motivating circumstances for considering now *functions'* $F(x)$ -**binomial** and $F(x)$ -**multinomial** coefficients. This is the case ($F(x) = \psi(x)$) for example in [49, 2004] **wherein we read:**

$$(x_1 + {}_{\psi}x_2 + {}_{\psi}\dots + {}_{\psi}x_k)^n = \sum_{\substack{s_1, \dots, s_k = 0 \\ s_1 + s_2 + \dots + s_k = n}}^n \binom{n}{s_1, \dots, s_k}_{\psi} x_1^{s_1} \dots x_k^{s_k}$$

where

$$\binom{n}{s_1, \dots, s_k}_{\psi} = \frac{n_{\psi}!}{(s_1)_{\psi}! \dots (s_k)_{\psi}!}.$$

Here above the *u-multinomial* number sequence formula from [5, 1936] by Morgan Ward is extended mnemonically to $\psi(x)$ function sequence definition of shifting x arguments formula written in Kwaśniewski upside-down notation (see for example [182, 2003], [183, 2001], [184, 2002], [185, 2005] for more ad this notation).

The above formulas introduced in case of number sequences in [5, 1936] by Morgan Ward were recalled from [5, 1936] by Alwyn F. Horadam and Anthony G. Shannon in [65, 1976] were the Ward Calculus of sequences framework (including umbral derivative and corresponding exponent) was used to enunciate two types of Ward's Staudt-Clausen theorems pertinent to this general calculus of number sequences.

The end of $\psi(x)$ -**multinomial Remark.**

Recently polynomial-fibonomial coefficients in [51, 2008] by Johann Cigler (i.e. x -Fibonomial coefficients from [50, 2001]) appear naturally in derivation of recurrence relations for powers of q -Fibonacci polynomials by Johann Cigler. Let us then come over to this multinomiality closing definition.

Definition 4 Let $F(x)$ be any natural, or complex numbers' non zero valued functions' sequence i.e. $F_n(x) \in \mathbb{N}$ or and $F_n(x) \in \mathbb{C}$. The $F(x)$ -**multinomial coefficient** is then identified with the symbol

$$(16) \quad \binom{n}{k_1, k_2, \dots, k_s}_{F(x)} = \frac{F_n(x)!}{F_{k_1}(x)! \cdot \dots \cdot F_{k_s}(x)!}$$

where $k_i \in \mathbb{N}$ and $\sum_{i=1}^s k_i = n$ for $i = 1, 2, \dots, s$. Otherwise it is equal to zero,

and where $F_r(x)! = F_r(x) \cdot F_{r-1}(x) \cdot \dots \cdot F_1(x)$.

Naturally for any natural n, k and $k_1 + \dots + k_m = n - k$ the following holds

$$(17) \quad \binom{n}{k}_{F(x)} \cdot \binom{n-k}{k_1, k_2, \dots, k_m}_{F(x)} = \binom{n}{k, k_1, k_2, \dots, k_m}_{F(x)},$$

$$\binom{n}{k_1, k_2, \dots, k_m}_{F(x)} = \binom{n}{k_1}_{F(x)} \binom{n-k_1}{k_2}_{F(x)} \cdots \binom{n-k_1-\dots-k_{m-1}}{k_m}_{F(x)}.$$

More ad Notation 2.

Does notation of items' representing - matter?

As a **Motto** for an experienced answer we propose:

Science is a language (from [205, 1996] by Doron Zeilberger);

Mathematical notation evolves like all languages (from [206, 1992] by Donald Ervin Knuth).

We shall use further on the traditional , XIX-th century rooted notation under presentation in spite of being inclined to quite younger notation from [46, 2009] by Bruce E. Sagan and Carla D. Savage. This wise, economic notation is ready for straightforward record of combinatorial interpretations and combinatorial interpretations' substantiation in terms of popular text book tiling model since long ago used for example to visualize recurrence for Fibonacci-like sequences ; see for example [152, 1989] by Ronald Graham, Donald Ervin Knuth, and Oren Patashnik. The translation from François Édouard Anatole Lucas via Dov Jarden and Theodor Motzkin notation [6, 1949] and notation of Bruce E. Sagan and Carla D. Savage [46, 2009] is based on the succeeding identifications: the symbol used for *U*-**binomials** is $\{...\}$ in place of $(...)_{\mathcal{U}}$. The would be symbol for *V*-**binomials** i.e. $P = \langle \dots \rangle$ in place of $(...)_{\mathcal{V}}$ is not considered at all in [46] while

$$\{n\} \equiv U_n \equiv n_{p,q}, \quad \langle n \rangle \equiv V_n.$$

In Bruce E. Sagan and Carla D. Savage notation we would then write down the fundamental and primordial sequences' binomial coefficients as follows.

Definition 5 Let $\{n\}$ be fundamental Lucas sequence as in [1, 1878] i.e $\{n\} \equiv U_n \equiv n_{p,q}$ then $\{n\}$ -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(18) \quad F\{n, k\} = \binom{\{n\}}{\{k\}}_{p,q} = \frac{\{n\}!}{\{k\}! \cdot \{n-k\}!} = \frac{\{n\}^k}{\{k\}!}$$

where $\{n\}! = \{n\} \cdot \{n-1\} \cdot \dots \cdot \{1\}$ and $\{n\}^k = \{n\} \cdot \{n-1\} \cdot \dots \cdot \{n-k+1\}$ and $F\{n, k\} = 0$ for $k > n$.

Definition 6 Let $\langle n \rangle$ be primordial Lucas sequence as in [1, 1878] i.e $\langle n \rangle \equiv V_n$ then $\langle n \rangle$ -binomial coefficients for any $n, k \in \mathbb{N} \cup \{0\}$ are defined as follows

$$(19) \quad P \langle n, k \rangle = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{p,q} = \frac{\langle n \rangle!}{\langle k \rangle! \cdot \langle n-k \rangle!} = \frac{\langle n \rangle^{\underline{k}}}{\langle k \rangle!},$$

where $\langle n \rangle! = \{n\} \cdot \{n-1\} \cdot \dots \cdot \{1\}$ and $P \langle n, k \rangle = 0$ for $k > n$. and $\{n\}^{\underline{k}} = \{n\} \cdot \{n-1\} \cdot \dots \cdot \{n-k+1\}$.

The above consequent symbols $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{p,q}$ and $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{p,q}$ are occasionally - in not exceptional conflict - with second kind Stirling numbers notation and Euler numbers notation respectively in the spirit of [152, 1989] what extends on both p, q - extensions' notation.

Regarding the symbol $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{p,q}$ one draws the attention of a reader to [11, 1967] where Verner Emil Hoggatt Jr. considers the U -binomial coefficients $\binom{n}{k}_U \equiv \left(\begin{matrix} n \\ k \end{matrix} \right)_{p,q}$ denoting them as $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ where $\{u_n\}_{n \geq 0} = U$ with U being the fundamental Lucas sequence. The author of [11, 1967] derives among others also recurrences [see this note formula (43)]

$$\begin{aligned} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= u_{k+1} \cdot \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + t \cdot u_{n-k-1} \cdot \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}, \\ \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= t \cdot u_{k-1} \cdot \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + u_{n-k+1} \cdot \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}, \end{aligned}$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{F(x)} = f(x)_{k+1} \cdot \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{f(x)} + 1 \cdot f(x)_{n-k-1} \cdot \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}_{F(x)},$$

where $F(x) = \langle f(x)_n \rangle_{n \geq 0}$ denotes the sequence of Fibonacci polynomials.

These recurrences are to be placed side by side with corresponding recurrence formulas in $\{\dots\}$ notation as for example the recurrence of the Theorem p.346 for $r = 2$ in [67, 1979]) by Anthony G. Shannon or the recurrence (10.3) in [17, 1989] by Ira M. Gessel and Xavier Gérard Viennot or with the Proposition 2.1 in [46, 2009] by Bruce E. Sagan and Carla D. Savage.

Anthony G. Shannon designates the recurrence of the Theorem p.346 in [67, 1979]) as the recurrence relation for multinomial coefficients

$$\binom{n}{s_1, s_2, \dots, s_r}_u = \frac{u_n^{(r)}!}{u_{s_1}^{(r)}! \cdot \dots \cdot u_{s_r}^{(r)}!},$$

where $u_n^{(r)} \equiv U_{r, n+r}^{(r)}$ and naturally $\sum_{k=1}^r s_k = n$.

For $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ corresponding notation see then also: [12, 1969] by Henri W. Gould, [65, 1976] by Alwyn F. Horadam and Anthony G. Shannon, [17, 1989] by Ira

M. Gessel and Xavier Gérard Viennot , [39, 2005], [40, 2005], [41, 2006] by Jaroslav Seibert and Pavel Trojovský and [42, 2007] by Pavel Trojovský.

Whereas as in the *subset-subspace problem* (Example [Ex. q^* ; 6] in subsection 4.3.) we rather need another natural notation. Namely for $q \neq 0$ introduce $q^* = \frac{p}{q}$ and observe that

$$\binom{n}{k}_U \equiv \binom{n}{k}_{p,q} = q^{k(n-k)} \binom{n}{k}_{1,q^*} \xrightarrow{q^* \mapsto 1} \binom{n}{k}.$$

The V -binomial $P \langle n, k \rangle = \langle \frac{n}{k} \rangle_{p,q} \equiv \binom{n}{k}_V$ is not considered in [46, 2010] - neither recurrences for $\binom{n}{k}_V$ coefficients are derived in earlier publications - up to knowledge of the present author .

3 $H(x)$ -binomial coefficients' recurrence

3.1. Let us recall convention resulting from (3).

Recall. The general solution of (3):

$$H(x) \equiv H(a(x), b(x); s(x), t(x)) = \langle H_n(x) \rangle_{n \geq 0}$$

is being called throughout this paper - **Ward-Horadam functions' sequence**.

From the characteristic equation of the recurrence (3)

$$(20) \quad z^2 - s(x) \cdot z - t(x) = 0$$

we readily see that for $H_0 = a(x)$, $H_1 = b(x)$ and $n \geq 0$,

$$(21) \quad H_n(x) \equiv H_n(a(x), b(x); p(x), q(x)) = A(x)p(x)^n + B(x)q(x)^n,$$

where $p(x), q(x)$ are roots of (20) and we have assumed that $p(x) \neq q(x)$ as well as that $\frac{p(x)}{q(x)}$ are not roots of unity (see for example Lemma 1 in [70, 1980] by Fritz Beukers). Naturally:

$$(22) \quad A(x) = \frac{b(x) - q(x)a(x)}{p(x) - q(x)}, \quad B = -\frac{b(x) - p(x)a(x)}{p(x) - q(x)}.$$

hence we may and we shall use the following conventional identifications-abbreviations

$$(23) \quad H(x) \equiv H(a(x), b(x); s(x), t(x)) \equiv H(A(x), B(x); s(x), t(x)).$$

As for the case $p(x) = q(x)$ included see for example Proposition 2.7 in [86, 2009].

Another explicit formula for Ward-Horadam functions sequences is the mnemonically extended formula (9) from [84, 2009] by Tugba Horzum and Emine Gökçen Kocer, where here down we use contractually the following abbreviations:

$$H_n(x) \equiv H_n(a(x), b(x); s(x), t(x)), \quad a(x) \equiv a, \quad b(x) \equiv b, \quad s(x) \equiv s \text{ and } t(x) \equiv t$$

$$(24) \quad H_n(x) = a \sum_{0 \leq k \leq \lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} s^{n-2k} t^k + \left(\frac{b}{s} - a \right) \sum_{0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} s^{n-2k} t^k$$

Note and compare. The recurrence (1.1) and (1.2) in [77, 1996] by Alwyn F. Horadam defines a polynomials' subclass of Ward-Horadam functions sequences defined by (3). The standard Jacques Binet form (1.8) in [77, 1996] of the recurrence (1.1) and (1.2) solution for Ward-Horadam polynomials sequences in [77, 1996] is the standard Jacques Binet form (19), (20) of the recurrence (3) solution for Ward-Horadam functions sequences.

The recurrence (2.23) in [86, 2009] by Tian-Xiao He and Peter Jau-Shyong Shiue defines exactly the class of Ward-Horadam functions' second order sequences and this paper standard Jacques Binet form (21), (22) of the recurrence (3) solution for Ward-Horadam functions sequences $H(x)$ constitutes the content of their Proposition 2.7. - as has been mentioned earlier. No recurrences for $H(x)$ -binomials neither for $H(x=1)$ -binomials are considered.

On Binet Formula - Historical Remark. We just quote Radoslav Rasko Jovanovic's information from

[http : //milan.milanovic.org/math/english/relations/relation1.html](http://milan.milanovic.org/math/english/relations/relation1.html) :

Quotation 1 *Binet's Fibonacci Number Formula was derived by Binet in 1843 although the result was known to Euler and to Daniel Bernoulli more than a century ago. ... It is interesting that A de Moivre (1667-1754) had written about Binet's Formula, in 1730, and had indeed found a method for finding formula for any general series of numbers formed in a similar way to the Fibonacci series.*

See also the book [154, 1989] by Steven Vajda.

3.2. The authors of [2] provide an easy proof of a general observation named there deservedly Theorem (Theorem 17) which extends automatically to the statement that the following recurrence holds for the general case of $\binom{r+s}{r,s}_{H(x)}$ $H(x)$ -binomial array in multinomial notation.

Theorem 1 *Let us admit shortly the abbreviations: $g_k(r, s)(x) = g_k(r, s)$, $k = 1, 2$. Let $s, r > 0$. Let $F(x)$ be any zero characteristic field nonzero valued functions' sequence ($F_n(x) \neq 0$). Then*

$$(25) \quad \binom{r+s}{r, s}_{F(x)} = g_1(r, s) \cdot \binom{r+s-1}{r-1, s}_{F(x)} + g_2(r, s) \cdot \binom{r+s-1}{r, s-1}_{F(x)}$$

where $\binom{r}{r, 0}_{F(x)} = \binom{s}{0, s}_{F(x)} = 1$ and

$$(26) \quad F(x)_{r+s} = g_1(r, s) \cdot F(x)_r + g_2(r, s) \cdot F(x)_s.$$

are equivalent.

On the way historical note Donald Ervin Knuth and Herbert Saul Wilf in [18, 1989] stated that Fibonomial coefficients and the recurrent relations for them appeared already in 1878 Lucas work (see: formula (58) in [1, 1878] p. 27 ; for U -binomials which "Fibonomials" are special case of). More over on this very p. 27 Lucas formulated a conclusion from his (58) formula which may be stated in notation of this paper formula (2) as follows: if $s, t \in \mathbb{Z}$ and $H_0 = 0$, $H_1 = 1$ then $H \equiv U$ and $\binom{n}{k}_U \equiv \binom{n}{k}_{n_p, q} \in \mathbb{Z}$.

Consult for that also the next century references: [153, 1910] by Paul Gustav Heinrich Bachmann or later on - [89, 1913] by Robert Daniel Carmichel [p. 40] or [6, 1949] by Dov Jarden and Theodor Motzkin where in all quoted positions it was also shown that $n_{p, q}$ - binomial coefficients are integers - for p and q representing distinct roots of this note characteristic equation (5).

Let us take an advantage to note that Lucas Théorie des Fonctions Numériques Simplement Périodiques i.e. investigation of exactly fundamental U and primordial V sequences constitutes the far more non-accidental context for binomial-type coefficients exhibiting their relevance at the same time to number theory and to hyperbolic trigonometry (in addition to [1, 1878] see for example [28], [29] and [31]).

It seems to be the right place now to underline that the *addition formulas* for Lucas sequences below with respective hyperbolic trigonometry formulas and also consequently U -binomials' recurrence formulas - stem from commutative ring R identity: $(x - y) \cdot (x + y) \equiv x^2 - y^2, x, y \in R$.

Indeed. Recall the characteristic equation notation: $z^2 = z^2 = s \cdot x + t$ as is common in many publications with the restriction: p, q roots are distinct. Recall that $p + q = s$ and $p \cdot q = -t$. Let $\Delta = s^2 - 4t$. Then $\Delta = (p - q)^2$. Hence we have as in [1, 1878]

$$(27) \quad 2 \cdot U_{r+s} = U_r V_s + U_s V_r, \quad 2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r.$$

Taking here into account the **U-addition formula** i.e. the first of two trigonometric-like L -addition formulas (42) from [1, 1878] ($L = U, V$) one readily recognizes that the U -binomial recurrence from the Corollary 18 in [2, 2009] is the

U -binomial recurrence (58) [1, 1878] which may be rewritten after François Édouard Anatole Lucas in multinomial notation and stated as follows: *according to the Theorem 2a below the following is true:*

$$2 \cdot U_{r+s} = U_r V_s + U_s V_r$$

is equivalent to

$$(28) \quad 2 \cdot \binom{r+s}{r, s}_{n_{p,q}} = V_s \cdot \binom{r+s-1}{r-1, s}_{n_{p,q}} + V_r \cdot \binom{r+s-1}{r, s-1}_{n_{p,q}}.$$

To this end see also Proposition 2.2. in [46, 2009] and compare it with both (28) and Example 3. below.

However there is no companion V -binomial recurrence i.e. for $\binom{r+s}{r, s}_V$ neither in [1, 1878] nor in [2, 2009] as well as all other quoted papers - up to knowledge of this note author.

Consequently then there is no overall $H(x)$ -binomial recurrence neither in [1, 1878] nor in [2] (2009) as well as all other quoted papers except for Final remark : p.5 in [48, 2010] up to knowledge of the present author.

The End of *the on the way historical note*.

The looked for $H(x)$ -binomial recurrence (29) accompanied by (30) might be then given right now in the form of (25) adapted to - Ward-Lucas functions' sequence case notation while keeping in mind that of course the expressions for $h_k(r, s)(x)$, $k = 1, 2$ below are designated by this $F(x) = H(x)$ choice and as a matter of fact are appointed by the recurrence (3).

For the sake of commodity to write down an observation to be next (named deservedly ? Theorem) let us admit shortly the abbreviations: $h_k(r, s)(x) = h_k(r, s) = h_k$, $k = 1, 2$. Then for $H(x)$ of the form (21) we evidently have what follows.

Theorem 2a.

$$(29) \quad \binom{r+s}{r, s}_{H(x)} = h_1(r, s) \binom{r+s-1}{r-1, s}_{H(x)} + h_2(r, s) \binom{r+s-1}{r, s-1}_{H(x)},$$

where $\binom{r}{r, 0}_{H(x)} = \binom{s}{0, s}_{H(x)} = 1$, *is equivalent to*

$$(30) \quad H_{r+s}(x) = h_1(r, s)H_r(x) + h_2(r, s)H_s(x).$$

where $H_n(x)$ is explicitly given by (21) and (22). **The end** of the Theorem 2a.

There might be various $h_1(r, s)(x) = h_1$ and $h_2(r, s)(x)_s = h_2$ solutions of (30) and (21). Compare (38) in Example 1 with (42) in Example 3 below.

As the possible $h_1(r, s)(x) = h_1$ and $h_2(r, s)(x)_s = h_2$ formal solutions of (30) and (21) we just may take

$$(31) \quad h_1(r, s)(x) = \frac{A(x) \cdot p(x)^{r+s}}{A(x) \cdot p^r + B(x) \cdot q(x)^r}, \quad h_2(r, r)(x) = \frac{B(x) \cdot q(x)^{r+s}}{A(x) \cdot p^s + B(x) \cdot q(x)^s}.$$

As another possible $h_1(r, s)(x)$ and $h_2(r, s)(x)_s = h_2$ solutions of (30) and (21) we may take: **for** $r \neq s$

$$(32) \quad h_1(r, s) \cdot (p((x)^r q(x)^s - q(x)^r p(x)^s) = p(x)^{r+s} q(x)^s - q(x)^{r+s} p(x)^s,$$

$$(33) \quad h_2(r, s) \cdot (q(x)^r p(x)^s - p(x)^r q(x)^s) = p(x)^{r+s} q(x)^r - q(x)^{r+s} p(x)^r.$$

while for $r = s$ apply formula (31) with $r = s$.

Usually the specific features of particular cases of (21) and (22) allow one to infer the particular form of (30) hence the form of $h_1(r, s)(x) = h_1$ and $h_2(r, s)(x)_s = h_2$.

As we soon shall see below (Theorem 2b), there is a specific way out from being bothered by this not illusory obstacle in V -binomials case. In general case one may always use the Georges Fontené recurrence [3, 1915] which however seems not to have combinatorial interpretation except for natural and 0 numbers n and their q -extensions $n_q = n_{q,q}$; $n \geq 0$. See Example 2 below.

3.3. Three special cases examples.

Example 1. This is a particular case of the Theorem 2a.

The recurrent relations (13) and (14) in Theorem 1 from [44, 2008] by Roberto Bagsarsa Corcino for $n_{p,q}$ -binomial coefficients are special cases of this paper formula (29) as well as of Th. 17 in [2] with straightforward identifications of g_1, g_2 in (13) and in (14) in [44] or in this paper recurrence (30) for $H(x = 1) = U[p, q]_n = n_{p,q}$ sequence. Namely, recall here now in multinomial notation this Theorem 1 from [44, 2008] by Roberto Bagsarsa Corcino:

$$(34) \quad \binom{r+s}{r, s}_{p,q} = q^r \binom{r+s-1}{r-1, s}_{p,q} + p^s \binom{r+s-1}{r, s-1}_{p,q},$$

$$(35) \quad \binom{r+s}{r, s}_{p,q} = p^r \binom{r+s-1}{r-1, s}_{p,q} + q^s \binom{r+s-1}{r, s-1}_{p,q},$$

which is equivalent to

$$(36) \quad (s+r)_{p,q} = p^s r_{p,q} + q^r s_{p,q} = (r+s)_{q,p} = p^r s_{p,q} + q^s r_{p,q},$$

what might be at once seen proved by noticing that

$$p^{r+s} - q^{r+s} \equiv p^s \cdot (p^r - q^r) + q^r \cdot (p^s - q^s).$$

Hence those mentioned straightforward identifications follow:

$$(37) \quad g_1 = q^r, \quad g_2 = p^s \text{ or } g_1 = p^r, \quad g_2 = q^s.$$

The recurrence (36) in Lucas notation reads

$$(38) \quad U_{s+r} = p^s U_r + q^r U_s = U_{r+s} = p^r U_s + q^s U_r.$$

Compare it with equivalent recurrence (42) from the **Example 3**. in order to notice that both h_1 and h_2 functions are different from case to case of recurrence (30) **equivalent realizations**.

Compare this example based on Theorem 1 in [44, 2008] by Roberto Corcino with with [47, 2008] **v[1]** by Maciej Dziemiańczuk (see there (1) and (2) formulas), and track as well - the simple combinatorial proof of the Corollary 3 in [47, 2009] **v[2]**) by Maciej Dziemiańczuk.

Example 2. This is a particular case of the Theorem 1.

Now let A be any natural numbers' or even complex numbers' valued sequence. One readily sees that also (1915 year from) Fontené recurrence for Fontené-Ward generalized A -binomial coefficients i.e. equivalent identities (6) , (7) in [12] **are special cases of** this paper formula (25) as well as of Th. 17 in [2] with straightforward identifications of h_1, h_2 in this paper formula (25) while this paper recurrence (26) becomes trivial identity.

Namely, the identities (6) and (7) from [12, 1969] read correspondingly:

$$(39) \quad \binom{r+s}{r, s}_A = 1 \cdot \binom{r+s-1}{r-1, s}_A + \frac{A_{r+s} - A_r}{A_s} \binom{r+s-1}{r, s-1}_A,$$

$$(40) \quad \binom{r+s}{r, s}_A = \frac{A_{r+s} - A_s}{A_r} \cdot \binom{r+s-1}{r-1, s}_A + 1 \cdot \binom{r+s-1}{r, s-1}_A,$$

where $p \neq q$ and $\binom{r}{r, 0}_L = \binom{s}{0, s}_L = 1$. And finally we have tautology identity

$$(41) \quad A_{s+r} \equiv \frac{A_{r+s} - A_s}{A_r} \cdot A_r + 1 \cdot A_s.$$

Example 2. becomes the general case of the Theorem 1. if we allow A to represent any zero characteristic field nonzero valued functions' sequence: $A = A(x) = \langle A_n(x) \rangle_{n \geq 0}$, $A_n(x) \neq 0$. In particular we may put $A_n(x) = H_n(x) \equiv H_n(s(x), t(x); A(x), B(x))$ to put into play also V -binomials; $V_n \equiv H_n(s, t; 1, -1)$.

Example 3. This is a particular case of the Theorem 2a.

The first example above is cognate to this third example in apparent way as might readily seen from François Édouard Anatole Lucas papers [1, 1878] or more recent article [121, 2001] by Hong Hu and Zhi-Wei Sun ; (see also $t = s$ case in [17, 1989] by Ira M. Gessel and Xavier Gérard Viennot on pp.23,24 .) In order to experience this let us start to consider now the number $H(x = 1) = U$ Lucas fundamental sequence *fulfilling* this note recurrence (2) with $U_0 = 0$ and $U_1 = 1$ as introduced in [1, 1878] and then - for example considered in [121, 2001]. There in [121, 2001] by Hong-Hu and Shi-Wei Sun - as a matter of fact - a kind of "pre-Theorem 17" from [2, 2009] is latent in the proof of Lemma 1 in [121]. We rewrite Lemma 1 by Hong-Hu and Shi-Wei Sun in multinomial notation and an arrangement convenient for our purpose here using sometimes abbreviation $U_n(p, q) \equiv U_n$.

(Note that the *addition formulas* for Lucas sequences hence consequently U -binomials' recurrence formulas [1, 1878] as well as $(p - q) \cdot (p^{j+k} - q^{j+k}) \equiv (p^{k+1} - q^{k+1}) \cdot (p^j - q^j) - p \cdot q(p^{j-1} - q^{j-1}) \cdot (p^k - q^k)$ - stem from commutative ring R identity: $(x - y) \cdot (x + y) \equiv x^2 - y^2$, $x, y \in R$.)

And so for $p \neq q$ and bearing in mind that $p \cdot q = -t$ - the following is true:

The identity (42) equivalent to

$$(p - q) \cdot (p^{j+k} - q^{j+k}) \equiv (p^{k+1} - q^{k+1}) \cdot (p^j - q^j) - p \cdot q(p^{j-1} - q^{j-1}) \cdot (p^k - q^k)$$

$$(42) \quad U_{j+k}(p, q) = U_{k+1} \cdot U_j(p, q) + t \cdot U_{j-1} \cdot U_k(p, q)$$

is equivalent to

$$(43) \quad \binom{j+k}{j, k}_U = U_{k+1} \cdot \binom{j+k-1}{j-1, k}_U + t \cdot U_{j-1} \cdot \binom{j+k-1}{j, k-1}_U,$$

where p, q are the roots of (5) and correspondingly the above Lucas fundamental sequence $H_n = U_n(p, q)$ i.e. $U_0 = 0$ and $U_1 = 1$ is given by its Binet form (6),(7).

Compare now (42) with equivalent this note recurrence (38) in order to notice that both h_1 and h_2 functions are different from case to case of recurrence (30) *equivalent realizations*.

Compare then this paper recurrence formula (42) with recurrence formula (4) in [46, 2009] or the recurrence (F) from [11, 1967] by Verner Emil Hoggatt, Jr. (see the end of this note Section 2) or $r = 2$ case of the the recursion for $u^{(r)}$ -multinomial coefficients (see Theorem on page 346) in [67, 1979] by Anthony G. Shannon (see this note Section 2).

Compare this paper recurrence formula (43) with Proposition 2.2. in [46, 2009] by Bruce E. Sagan and Carla D. Savage.

Compare this paper recurrence (28) equivalent to (5) and proposition 2.2. in [46, 2009] and note that (5) in [46, 2010] is just the same - as (58) in [1, 1878] - the same except for notation. The translation from "younger" notation of Bruce E. Sagan and Carla D. Savage (from one - left hand - side) into more matured by tradition notation of François Édouard Anatole Lucas (from the other - right hand - side) is based on the identifications: the symbol used for U -binomials is $\{\dots\}$ in place of $(\dots)_U$ and

$$\{n\} \equiv U_n \equiv n_{p,q}, \langle n \rangle \equiv V_n.$$

For $s = t = 1$ we get Fibonacci $U_n = F_n$ sequence with recurrence (41) becoming the recurrence known from Donald Ervin Knuth and Herbert Saul Wilf masterpiece [18, 1989].

Example 3. becomes **more general** case of the Theorem 2. if we allow U to represent any zero characteristic field nonzero valued **functions'** sequence $U(x) = \langle U_n(x) \rangle_{n \geq 0}$, $U_n(x) = \frac{p(x)^n - q(x)^n}{p(x) - q(x)} \equiv n_{p(x),q(x)}, p(x) \neq q(x)$ i.e. $p(x), q(x)$ denote distinct roots of (20) and we have assumed as well that $\frac{p(x)}{q(x)}$ are not roots of unity.

The End of three examples.

Let us now come back to consider the case of V -binomials recurrence according to what after Theorem 2a was called - "a not illusory obstacle". For that to do carefully let us at first make precise the main item. $H(x) = \langle H_n(x) \rangle_{n \geq 0}$ is considered as a solution of **second** order recurrence (3) with peculiar case of (3) becoming the first order recurrence **excluded**. This is equivalent to say that $0 \neq p \neq q \neq 0$ and $A \neq 0 \neq B$ (compare with [63, 1974] by Anthony J. W. Hilton), where for notation convenience we shall again use awhile shortcuts for (3):

$$a(x) \equiv a, b(x) \equiv b, s(x) \equiv s, t(x) \equiv t,$$

$$p(x) \cdot q(x) = -t(x) \equiv p \cdot q = -t,$$

$$H(x) = H(A(x), B(x), p(x), q(x)) = H(A, B) = A \cdot H(1, 1;),$$

$$H(1, 1) = V(x) = V = (p-q)U(1, -1), U = U(x) = U(1, 1), U(A, B) = U(A, B, p, q)$$

referring to this note (23) and the next to (23) abbreviations:

$$\begin{aligned} H(x) &\equiv H(a(x), b(x); s(x), t(x)) \equiv H(A(x), B(x); s(x), t(x)), \\ H_n(x) &\equiv H_n(a(x), b(x); s(x), t(x)), \\ H_n(x) &= H_n(A, B) = A \cdot p^n + B \cdot q^n \quad H_n(1, 1) = V_n(x) = V_n. \end{aligned}$$

Now although $H(A, B) = (p - q) \cdot U(A, -B)$ the corresponding recurrences are different. Recall and then compare corresponding recurrences:

The identity

$$(p - q) \cdot (p^{r+s} - q^{r+s}) \equiv (p^{s+1} - q^{s+1}) \cdot (p^r - q^r) - p \cdot q(p^{r-1} - q^{r-1}) \cdot (p^s - q^s)$$

due to $p \cdot q = -t$ is *equivalent to*

$$U_{r+s}(p, q) = U_{s+1} \cdot U_r(p, q) + t \cdot U_{r-1} \cdot U_s(p, q)$$

(combinatorial derivation - see [130, 1999], [131, 1999]). This recurrence in its turn *is equivalent to*

$$\binom{r+s}{r, s}_U = U_{s+1} \cdot \binom{r+s-1}{r-1, s}_U + t \cdot U_{r-1} \cdot \binom{r+s-1}{r, s-1}_U.$$

Similarly *the identity*

$$(p - q) \cdot (p^{r+s} + q^{r+s}) \equiv (p^{s+1} - q^{s+1}) \cdot (p^r + q^r) - p \cdot q(p^{r-1} + q^{r-1}) \cdot (p^s - q^s)$$

due to $p \cdot q = -t$ is *equivalent to*

$$(44) \quad V_{r+s}(p, q) = U_{s+1} \cdot V_r(p, q) + t \cdot V_{r-1} \cdot U_s(p, q)$$

For combinatorial interpretation derivation of the above for Fibonacci and Lucas sequences ($t = 1 = s$) see [130, 1999], [131, 1999] and then see for more [132, 2003] also by Arthur T. Benjamin and Jennifer J. Quinn.

Definition 7 Let $\{n(x)\} \equiv U_n$. $\langle n(x) \rangle \equiv V_n = p^n(x) + q^n(x)$, hence $V_0 = 2$ and $V_1 = p + q = s(x)$. Let $(p(x) - q(x)) \cdot U_n = p^n(x) - q^n(x)$, hence $U_0 = 0$ and $V_1 = 1$; (roots are distinct). Then V -mixed- U binomial coefficients for any $r, s \in \mathbb{N} \cup \{0\}$; are defined as follows

$$(45) \quad \binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} = \frac{V_{r+s}!}{V_r! \cdot U_s!},$$

$$\binom{n}{k}_{\langle \cdot \rangle / \{ \cdot \}} = 0 \text{ for } k > n \text{ and } \binom{n}{0}_{\langle \cdot \rangle / \{ \cdot \}} = 1.$$

Note that: $\binom{r+s}{r, s}_{\langle \cdot \rangle / \{ \cdot \}} \neq \binom{r+s}{s, r}_{\langle \cdot \rangle / \{ \cdot \}}.$

Note that: $\binom{r+s}{r,s}_{\langle \cdot \rangle / \{ \cdot \}} \neq \binom{r+s}{r,s}_{\{ \cdot \} / \langle \cdot \rangle}$

Theorem 2b. The recurrence (44) is equivalent to

$$(46) \quad \binom{r+s}{r,s}_{\langle \cdot \rangle / \{ \cdot \}} = U_{s+1} \cdot \binom{r+s-1}{r-1,s}_{\langle \cdot \rangle / \{ \cdot \}} + t \cdot U_s \cdot \binom{r+s-1}{r,s-1}_{\langle \cdot \rangle / \{ \cdot \}}.$$

Example 4. Application of Theorems 1, 2a and the Theorem 2b method. Recall the characteristic equation notation: $z^2 = z^2 = s \cdot x + t$ as is common in many publications with the restriction: p, q roots are distinct. Recall that $p + q = s$ and $p \cdot q = -t$. Let $\Delta = s^2 - 4t$. Then $\Delta = (p - q)^2$. Hence we have as in [1, 1878]

$$2 \cdot U_{r+s} = U_r V_s + U_s V_r, \quad 2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r.$$

Consider **V-addition formula** i.e. the second of two trigonometric-like addition formulas (42) from [1, 1878]. One readily recognizes that *according to the Theorem 2b the following is true*: $2 \cdot V_{r+s} = V_r V_s + \Delta \cdot U_s U_r$ is equivalent to

$$(47) \quad 2 \cdot \binom{r+s}{r,s}_{\langle \cdot \rangle / \{ \cdot \}} = V_s \cdot \binom{r+s-1}{r-1,s}_{\langle \cdot \rangle / \{ \cdot \}} + \Delta \cdot U_r \cdot \binom{r+s-1}{r,s-1}_{\langle \cdot \rangle / \{ \cdot \}}.$$

The combinatorial interpretation of mixed binomials is part of the subject of a forthcoming "in statu nascendi" note.

4 Snatchy information on combinatorial interpretations of $H(x=1)$ -binomials and their relatives.

4.1. In regard to **combinatorial interpretations** of H -binomial or F -multinomial coefficients or related arrays we leave that subject apart from this note. Nevertheless we direct the reader to some comprise papers and references therein via listing; these are here *for example* the following:

Listing. **1.** [15, 1984] by Bernd Voigt: on common generalization of binomial coefficients, Stirling numbers and Gaussian coefficients .

Listing. **2.** [19, 1991] by Michelle L. Wachs and Dennis White and in [23, 1994] by Michelle L. Wachs: on p, q -Stirling numbers and set partitions.

Listing. **3.** [22, 1993] by Anne De Médicis and Pierre Leroux: on Generalized Stirling Numbers, Convolution Formulae and (p, q) -Analogues.

Listing. **4.** [127, 1998] John Konvalina: on generalized binomial coefficients and the Subset-Subspace Problem. Consult examples [Ex. q^* ; 6] and [Ex. q^*

; 7] in 4.3. below. Then see also the article [128, 2000] by John Konvalina on an unified simultaneous interpretation of binomial coefficients of both kinds, Stirling numbers of both kinds and Gaussian binomial coefficients of both kinds.

Listing. 5. Ira M. Gessel and Xavier Gérard Viennot in [17, 1989] deliver now the well known their interpretation of the fibonomials in terms of non-intersecting lattice paths.

Listing. 6. In [37, 2004] Jeffrey Brian Remmel and Michelle L. Wachs derive a new rook theory interpretation of a certain class of generalized Stirling numbers and their (p, q) -analogues. In particular they prove that their (p, q) -analogues of the generalized Stirling numbers of the second kind may be interpreted in terms of colored set partitions and colored restricted growth functions.

Listing. In 7. [129, 2005] by Ottavio M. D'Antona and Emanuele Munarini deal - in terms of weighted binary paths - with combinatorial interpretation of the connection constants which is in particular unified, simultaneous combinatorial interpretation for Gaussian coefficients, Lagrange sum, Lah numbers, , q -Lah numbers, Stirling numbers of both kinds , q -Stirling numbers of both kinds. Note the usefull correspondence: weighted binary paths \Leftrightarrow edge colored binary paths

Listing. 8. Maciej Dziemiańczuk in [155, 2011] extends the results of John Konvalina from 4. above. The Dziemiańczuk' ζ - analogues of the Stirling numbers arrays of both kinds cover ordinary binomial and Gaussian coefficients, p, q -Stirling numbers and other combinatorial numbers studied with the help of object selection, Ferrers diagrams and rook theory. The p, q -**binomial** arrays are special cases of ζ - numbers' arrays, too.

ζ -number of the first and the second kind is the number of ways to select k objects from k of n boxes without box repetition allowed and with box repetition allowed, respectively. The weight vectors used for objects constructions and statements derivation are functions of parameter ζ .

Listing. 9. As regards combinatorial interpretations via tilings in [132, 2003] and [133, 2010] - see 4.2. below.

Listing. 10. In [81, 2003] Johann Cigler introduces "abstract Fibonacci polynomials" - interpreted in terms of Morse coding sequences monoid with concatenation (monominos and dominos tiling then). Cigler's abstract Fibonacci polynomials are monoid algebra over reals valued polynomials with straightforward Morse sequences i.e. tiling recurrence originated (1.6) "addition formula"

$$F_{m+n}(a, b) = F_{m+1}(a, b) \cdot F_m(a, b) + b \cdot F_{n-1}(a, b) \cdot F_n(a, b),$$

which is attractive and seductive to deal with within the context of this paper Theorem 1. The combinatorial tiling interpretation of the model is its construction framed in the Morse coding sequences monoid with concatenation (monominos and dominos tiling then).

Listing. **11**....to be juxtaposed to Listing. **12**..

In [136, 2009] Johann Cigler considers special Ward-Horadam *polynomials sequences* and reveals the tiling combinatorial interpretation of these special Ward-Horadam *polynomials sequences* in the spirit of Morse with monomino, domino alphabet monoid as here above in Listing. **10**.. Compare with technique in [204, 1994] by Dominique Foata and Guo Niu Han **where** one considers "de rubans d'ordre m , qu'on peut représenter comme m rubans remplis de monominos et de dominos, de même longueur n " and where Pafnuty Lvovich Chebyshev polynomials are treated.

1. In [136, 2009] by Johann Cigler the q -Fibonacci polynomial $F_n(x, s, q) = \sum_{c \in \Phi_n} w(c) \equiv w(\Phi_n)$ is the q -weight function of the set Φ_n of all words (coverings) c of length $n-1$ in Morse (tiling) alphabet $\{a, b\}$ i.e. it is the corresponding generation function for number of linear **q -weighted** tilings as clearly Φ_n may be identified with the set of all linear **q -weighted** tilings of $(n-1) \times 1$ rectangle or equivalently with Morse code sequences of length $n-1$.

Then classical Fibonacci polynomials $F_n(x, s, 1)$ satisfy this paper recursion (3) i.e the Ward-Horadam functions' sequence recursion below - with $H_0(x) = 0$, $H_1(x) = x$; $s(x) = x$ and $t(x) = s$; (here $H \mapsto F$):

$$H_{n+2}(x) = s(x) \cdot H_{n+1}(x) + t(x) \cdot H_n, \quad n \geq 0, \quad H_0 = a(x), H_1 = b(x).$$

The $F(x, s, q)$ -binomial array $\binom{n}{k}_{F(x,s,q)}$ is not considered in [136, 2009]. Similarly:

2. the q -Lucas polynomial $L_n(x, s, q) = \sum_{c \in \Lambda_n} w(c) \equiv w(\Lambda)$ is the q -weight function of the set Λ_n of all coverings c with arc monominos and dominos of the circle whose circumference has length n . Hence $L_n(x, s, q)$ is corresponding generation function for number of **q -weighted** tilings of the circle whose circumference has length n . It may be then combinatorially seen that $w(\Lambda_n) = w(\Phi_{n+1}) + s \cdot w(\Phi_{n-1})$ hence specifically for the classical Fibonacci and Lucas polynomials we get the well known relation $L_n(x, s, 1) = F_{n+1}(x, s, 1) + s \cdot F_{n-1}(x, s, 1)$. In [136, 2009] Johann Cigler proves more: $L_n(x, s, q) = F_{n+1}(x, s, q) + s \cdot F_{n-1}(x, s, q)$ and provides a combinatorial interpretation of this relation, too.

Polynomials $L_n(x, s, 1)$ satisfy this paper recursion (3) with $H_0(x) = 2$, $H_1(x) = x$; $s(x) = x$ and $t(x) = s$; (here $H \mapsto L$)

The $L(x, s, q)$ -binomial array $\binom{n}{k}_{L(x,s,q)}$ is not considered in [136, 2009].

Listing. **12**. ..to be juxtaposed to Listing. **11**..

[46, 2009] by Bruce E. Sagan and Carla D. Savage the symbol $\{n\} \equiv U_n$ denotes the n -th element of the fundamental Lucas sequence U satisfying this paper recurrence (2) with initial conditions $\{0\} = 0$, $\{1\} = 1$. Naturally $\{n\}$ is a polynomial in parameters s, t . So is also the U -binomial coefficient $\binom{n}{k}_U \equiv \binom{n}{k}_{p,q}$.

Similarly - the symbol $\langle n \rangle \equiv V_n$ denotes the n -th element of the primordial Lucas sequence V satisfying this paper recurrence (2) with initial conditions $\langle 0 \rangle = 2$, $\langle 1 \rangle = s$. Naturally $\langle n \rangle$ is a polynomial in parameters s, t . So is also the V -binomial coefficient $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_V \equiv \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle_{p,q}$. V -binomials are not considered in [46, 2009]. Both fundamental and primordial sequences are interpreted via tilings similarly to the above in **11**. Johann Cigler attitude rooted in already text-books tradition - see for example [152, 1989] by Ronald Graham, Donald Ervin Knuth and Oren Patashnik.

An so: $\{n\}$ is generation function for number of linear tilings of $(n-1) \times 1$ rectangle or equivalently of number of Morse code sequences of length $n-1$.

$\langle n \rangle$ is generation function for number of circular tilings of the circle whose circumference has length n . Using naturally proved (just seen) relations Bruce E. Sagan and Carla D. Savage derive **two** combinatorial interpretations of the the same $\left\{ \begin{smallmatrix} m+n \\ m,n \end{smallmatrix} \right\}_{p,q}$ via Theorem 3.1. from which we infer the following.

1. $\left\{ \begin{smallmatrix} m+n \\ m,n \end{smallmatrix} \right\}_{p,q}$ is the weight of all linear tilings of all integer partitions λ inside the $m \cdot n$ rectangle

hence $\left\{ \begin{smallmatrix} m+n \\ m,n \end{smallmatrix} \right\}_{p,q}$ is the generating function for numbers of such tilings of partitions.

2. $2^{m+n} \cdot \left\{ \begin{smallmatrix} m+n \\ m,n \end{smallmatrix} \right\}_{p,q}$ is the weight of all circular tilings of all integer partitions λ inside the $m \cdot n$ rectangle

hence $\left\{ \begin{smallmatrix} m+n \\ m,n \end{smallmatrix} \right\}_{p,q}$ is the generating function for numbers of such tilings of partitions.

Explanation. from [46, 2009]. A *linear tiling of a partition* λ is a covering of its Ferrers diagram with disjoint dominos and monominos obtained by linearly tiling each λ_i part. In circular tiling of a partition λ one performs circular tiling of each λ_i part

Listing. **13.** In [177, 2009] *Counting Bipartite, k -Colored and Directed Acyclic Multi Graphs Through F -binomial coefficients* by Maciej Dziemiańczuk, the explicit relation between the number of the so called in [177, 2009] k -colored α -multigraphs and $N(\alpha)$ -multinomial coefficients has been established.

Here it is a directed acyclic graph with α -multiple edges (i.e. any two vertices might be connected by at most $(\alpha-1)$ directed edges) what is called acyclic α -multi digraph.

Let $N(\alpha) = \{n_{\alpha,\alpha}\}_{n \geq 0}$ i.e. one considers the $p = q \equiv \alpha$ case of $U = \langle n_{p,q} \rangle$ number sequence, hence $n_{\alpha,\alpha} = n \cdot \alpha^{n-1}$. For example (we quote these from [177, 2009]):

1. $N(1) = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots$
2. $N(2) = 0, 1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, \dots$

3. $N(3) = 0, 1, 6, 27, 108, 405, 1458, 5103, 17496, 59049, 196830, \dots$
4. $N(4) = 0, 1, 8, 48, 256, 1280, 6144, 28672, 131072, 589824, 2621440, \dots$

Maciej Dziemiańczuk proves in [177, 2009] that **if** $\gamma_{\alpha,n,k}$ is the number of all k -colored α -multigraphs with n vertices, **then**

$$\gamma_{\alpha,n,k} = \sum_{\substack{b_1 + \dots + b_k = n \\ b_1, \dots, b_k \geq 0}} \binom{n}{b_1, b_2, \dots, b_k}_{N(\alpha)}$$

The case of k -colored graphs *without multiple edges* i.e. $\alpha = 2$ - is to be found in [179, 2003] by Steven R. Finch.

Maciej Dziemiańczuk in [177, 2009] proves also that the unsigned values of the first row of inversion matrix for $N(\alpha)$ -binomial coefficients considered there are equal to the numbers of directed acyclic α -multigraphs with n nodes.

Ad Notation Remark. The author of [177, 2009] and [178, 2008] - uses misleading, cumbersome notation. He writes $m_{n,k}^{-1} = \binom{n}{k}_F^{-1}$ instead of $M^{-1}_{n,k} = \left(\binom{r}{s}_F \right)^{-1}_{n,k}$, where here $M = (m_{n,k})$.

The exemplification that now follows is concerned with the referring to Steven R. Finch particular result from [177, 2009]. For that it might be helpful to consult eventually also ψ -**mulinomial remark** in this note Section 2.

In view of the final Remark in [180, 2004], the fundamental logarithmic Fib-Binomial Formula ([180], Section 4)

$$\phi_n^{(t)}(x +_F a) \equiv \left[\exp\{a \partial_F\} \phi_n^{(t)} \right] (x) = \sum_{k \geq 0} \left[\begin{matrix} n \\ k \end{matrix} \right]_F \phi_{n-k}^{(t)}(a) x^k$$

$$t = 0, 1; \quad |x| < a; \quad n \in \mathbb{Z}$$

may be considered as F -Binomial formula for any natural numbers valued sequence F with $F_0 = 1$ (the class considered in [180, 2004] is much broader).

Now put in F -binomial formula above $t = 0$ and $a = x = 1$ and pay attention to - that according to the Steven M. Roman definition of hybrid binomial coefficients in [181, 1992] one states - this time for F -hybrid binomial coefficients - the wanted for $n, k \geq 0$ identification

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_F = \binom{n}{k}_F.$$

For special F -sequences introduced as F -cobweb posets admissible sequences ([182, 2003], [183, 2001], [185, 2005]) the F -binomial coefficients for $n, k \geq 0$ acquire joint combinatorial interpretation from Listing. **14.**

Then in Morgan Ward Calculus of Sequences ([5, 1936]) notation adapted in umbra calculus by Kwaśniewski one may write the following appealing formula :

$$(1 +_F 1)^n \equiv \sum_{n \geq 0} \binom{n}{k}_F$$

which is a special case of

$$(1 +_F \dots +_F 1)^n \equiv \sum_{n \geq 0} \binom{n}{k}_F.$$

In particular application (see [177, 2009] by Maciej Dziemiańczuk) we may then write in Ward Morgan [5, 1936] spirit notation that the number $\gamma_{n,2}$ of all 2-colored graphs as in [179] is now due to [177, 2009] equal to

$$\gamma_{n,2} = \sum_{k \geq 0} \binom{n}{k} \cdot 2^{k(n-k)} = \sum_{k \geq 0} \binom{n}{k}_{N(2)} \equiv (1 +_{N(2)} 1)^n$$

For more - consult [177, 2009] by Maciej Dziemiańczuk.

Listing. 14. Cobweb posets' partitions and hyper-boxes tilings.

14.1 A definition of the cobweb poset.

The cobweb posets were introduced under this name in several paper - see: [196, 2007] , [186]-[191] , [195, 2008] and references therein. There - the cobweb posets were defined in terms of their poset Hasse diagrams. The definition we are here delivering is taken from [187, 2009], [186, 2009] (for equivalent definitions see also: [193, 2009], [194, 2009], [192, 2009]), [187, 2010]).

Namely; let us consider any infinite chain $\{\Phi_k\}_{k \geq 0}$ of trivial posets \equiv antichains \equiv a chain of independent sets \equiv a chain of trivial unary relations, then we define

Definition 8 (cobweb poset Π_n) Let $n \in N \cup \{0\} \cup \{\infty\}$. Then

$$\Pi_n = \oplus_{s=0}^n \Phi_s,$$

$$\Pi = \oplus_{s \geq 0} \Phi_s,$$

where \oplus denotes ordinal sum of posets.

Recall that the ordinal sum [linear sum] of two disjoint ordered sets P and Q , denoted by $P \oplus Q$, is the union of P and Q , with P 's elements ordered as in P while Q 's elements are correspondingly ordered as in Q , and for each $x \in P$ and $y \in Q$ we put $x \leq y$. The Hasse diagram of $P \oplus Q$ we construct placing Q 's diagram just above P 's diagram and with an edge between each minimal element of Q and each maximal element of P .

The cobweb posets might be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs; see for example [187, 2009], [186, 2009], [193, 2009], [195, 2008]. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs in di-bicliques of the complete relations chain.

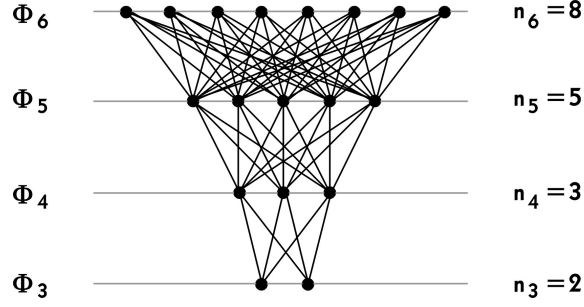


Figure 1: Display of four levels of Fibonacci numbers' finite Cobweb sub-poset

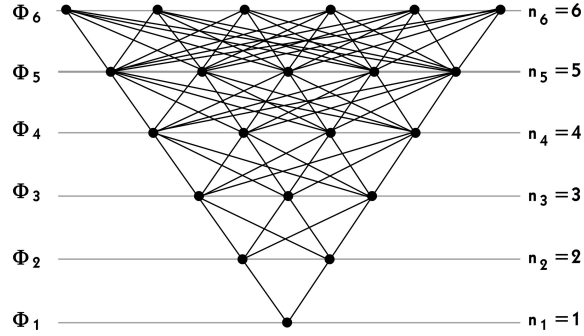


Figure 2: Display of Natural numbers' finite Cobweb sub-poset

14.2 Combinatorial interpretation of F -binomials in terms of cobweb poset.

Combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) called KoDAGs has been developed in [193, 2009], [191, 2010], [187, 2009], [186, 2009], [188, 2010], [189, 2010]. The recent equivalent formulation of this combinatorial interpretation is to be found also in [192, 2009] or [194, 2009] from which we quote it here down. We shall now start to use the upside down notation : $F_n \equiv n_F$ as an extension of $n_{p,q} \equiv U_n(p, q)$ or specifically $n_{1,q} \equiv U_n(1, q) \equiv n_q$ notation.

Definition 9 *Admissible sequence F -binomial coefficients are defined as fol-*

lows

$$\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)_F!} = \frac{n_F \cdot (n-1)_F \cdot \dots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \dots \cdot k_F} = \frac{n_F^k}{k_F!}$$

while $n, k \in \mathbb{N}$ and $0_F! = n_F^0 = 1$ with $n_F^k \equiv \frac{n_F!}{k_F!}$ staying for falling factorial. Zero or natural numbers' valued F sequence is called F -graded poset **admissible** sequence iff $\binom{n}{k}_F \in \mathbb{N} \cup \{0\}$ (In particular we shall use the expression - F -cobweb admissible sequence).

Definition 10

$$C_{max}(\Pi_n) \equiv \{c = \langle x_0, x_1, \dots, x_n \rangle, x_s \in \Phi_s, s = 0, \dots, n\}$$

i.e. $C_{max}(\Pi_n)$ is the set of all maximal chains of Π_n

and consequently (see Section 2 in [197, 2011] on Cobweb posets' coding via N^∞ lattice boxes)

Definition 11 ($C_{max}^{k,n}$)

$$C_{max}(\Phi_k \rightarrow \Phi_n) \equiv \{c = \langle x_k, x_{k+1}, \dots, x_n \rangle, x_s \in \Phi_s, s = k, \dots, n\} \equiv$$

$$\equiv \{\text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle\} \equiv C_{max}(\langle \Phi_k \rightarrow \Phi_n \rangle) \equiv C_{max}^{k,n}.$$

Note. The $C_{max}(\Phi_k \rightarrow \Phi_n) \equiv C_{max}^{k,n}$ is the hyper-box points' set [197], [186] of Hasse sub-diagram corresponding maximal chains and it defines biunivoquely the layer $\langle \Phi_k \rightarrow \Phi_n \rangle = \bigcup_{s=k}^n \Phi_s$ as the set of maximal chains' nodes (and vice versa) - for these arbitrary F -denominated **graded** DAGs (KoDAGs included).

The formulation of the fractals reminiscent combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following observation (named deservedly ? a Theorem) equivalent to that of [193], [192], [194].

Theorem 3 [196], [194], [193], [188]

For F -cobweb admissible sequences F -binomial coefficient $\binom{n}{k}_F$ is the cardinality of the family of equipotent to $C_{max}(P_m)$ mutually disjoint maximal chains sets, all together **partitioning** the set of maximal chains $C_{max}(\Phi_{k+1} \rightarrow \Phi_n)$ of the layer $\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$, where $m = n - k$.

Comment 1. For the above combinatorial interpretation of F -binomials' array it does not matter of course whether the diagram is being directed or not, as this combinatorial interpretation is equally valid for partitions of the family of

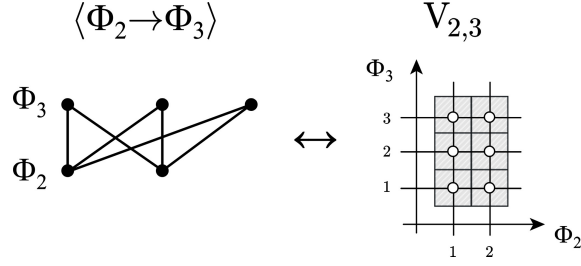


Figure 3: Bipartite layer $\langle \Phi_3 \rightarrow \Phi_4 \rangle$ with six maximal chains and equivalent hyper-box $V_{2,3}$ with six white circle-dots

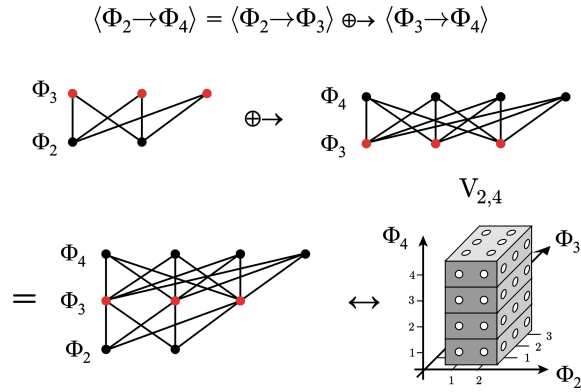


Figure 4: Display of the natural join of bipartite layers $\langle \Phi_k \rightarrow \Phi_{k+1} \rangle$ $F = N$, resulting in $2 \cdot 3 \cdot 4$ maximal chains and equivalent hyper-box $V_{2,4}$ with $2 \cdot 3 \cdot 4$ white circle-dots.

$SimplePath_{max}(\Phi_k - \Phi_n)$ in comparability graph of the Hasse digraph with self-explanatory notation used on the way. The other insight into this irrelevance for combinatoric interpretation is in [194]: colligate the coding of $C_{max}^{k,n}$ by hyper-boxes.

14.3 Combinatorial interpretation of F -binomials in discrete hyper-boxes language.

In order to formulate the above combinatorial interpretation let us recall indispensable identifications from [188, 2009], [194, 2009].

$C_{max}(\Pi_n)$ is the set of all maximal chains of Π_n .

$$C_{max}^{k,n} = \{ \text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle \}.$$

Consult Section 3. in [197, 2011] in order to view $C_{max}(\Pi_n)$ or $C_{max}^{k,n}$ as the hyper-boxes of points.

Now following [188, 2009], [186, 2009]), [187, 2009], [197, 2011]) - we define

the F -hyper-box $V_{k,n}$.

Definition 12 *The the discrete finite rectangular F -hyper-box or $(k, n) - F$ -hyper-box or in everyday parlance just (k, n) -box $V_{k,n}$ is the Cartesian product*

$$V_{k,n} = [k_F] \times [(k+1)_F] \times \dots \times [n_F].$$

We identify the following two object just by agreement according to the F -natural identification:

$$C_{max}^{k,n} \equiv V_{k,n}$$

i.e.

$$C_{max}^{k,n} = \{ \text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle \} \equiv V_{k,n}.$$

For numerous illustrations via natural join of posets alike that by Fig.4. - see [186, 2009]), [187, 2009] and [197, 2011].

Now in discrete hyper-boxes language the *combinatorial interpretation reads*:

Theorem 4. [187, 2009]

*For F -cobweb admissible sequences F -binomial coefficient $\binom{n}{k}_F$ is the cardinality of the family of equipotent to $V_{0,m}$ mutually disjoint discrete hyper-boxes, all together **partitioning** the discrete hyper-box $V_{k+1,n} \equiv$ the layer $\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$, where $m = n - k$.*

14.4 The cobweb tiling problem in the language of discrete hyper-boxes.

Comment 2. General "fractal-reminiscent" comment. The discrete m -dimensional F -box ($m = n - k$) with edges' sizes designated by natural numbers' valued sequence F where invented in [197, 2010] as a response to the so called *cobweb tiling problem* posed in [196, 2007] and then repeated in [193, 2009]. This tiling problem was considered by Maciej Dziemiańczuk in [198, 2008] where it was shown that not all admissible F -sequences permit tiling as defined in [196, 2007]. Then - after [197, 2009] this tiling problem was considered by Maciej Dziemiańczuk in discrete hyper-boxes language [199, 2009].

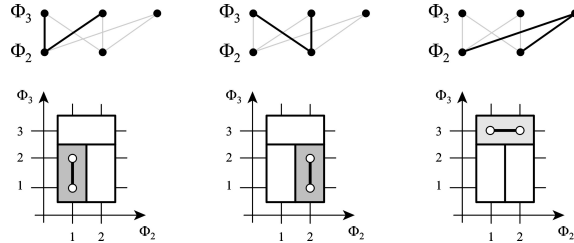


Figure 5: Correspondence between tiling of F -box $V_{3,4}$ with two white-dots boxes and tiling of the $\langle \Phi_3 \rightarrow \Phi_4 \rangle$ with two-chain subposets.

Recall the fact ([196, 2007], [193, 2009]): Let F be an admissible sequence. Take any natural numbers n, m such that $n \geq m$, then the value of F -binomial coefficient $\binom{n}{m}_F$ is equal to the number of sub-boxes that constitute a κ -partition of m -dimensional F -box $V_{m,n}$ where $\kappa = |V_m|$.

Definition 13 Let $V_{m,n}$ be a m -dimensional F -box. Then any κ -partition into sub-boxes of the form V_m is called tiling of $V_{m,n}$.

Hence **only these** partitions of m -dimensional box $V_{m,n}$ are admitted for which all sub-boxes **are of the form** V_m i.e. we have a kind of **self-similarity**.

It was shown in [198, 2008] by Maciej Dziemiańczuk that the only admissibility condition is not sufficient for the existence a tiling for any given m -dimensional box $V_{k,n}$. Kwaśniewski in [196, 2007] and [193, 2009] posed the question called *Cobweb Tiling Problem* which we repeat here.

Tiling problem

Suppose that F is an admissible sequence. Under which conditions any F -box $V_{m,n}$ designated by sequence F has a tiling? Find effective characterizations and/or find an algorithm to produce these tilings.

In [199, 2009] by Maciej Dziemiańczuk one proves the existence of such tiling for certain sub-family of admissible sequences F . These include among others $F = \text{Natural numbers}$, $F = \text{Fibonacci numbers}$, or $F = \langle n_q \rangle_{n \geq 0}$ Gaussian sequence. Original extension of the above tiling problem onto the general case multi F -multinomial coefficients is proposed in [199, 2009], too. Moreover - a reformulation of the present cobweb tiling problem into a clique problem of a graph specially invented for that purpose - is invented.

Listing. 15. Fences and zigzagged roots.

The tiling interpretation of Fibonacci numbers is known since decades; note for example picture-symbolic language in Chapter 7 from [152, 1989] by Ronald Graham, Donald Ervin Knuth and Oren Patashnik. See this picturesque symbolic technique in relevant creative application by Dominique Foata and Guo Niu Han in [204, 1994].

For *example 0*: F_{n+1} equals to the number of compositions of n into **1**'s and **2**'s; see [200, 1969] by Verner Emil Hoggatt, Jr. and D.A. Lind and for more [201, 1975] by Krithnaswami Alladi and Verner Emil Hoggatt, Jr. while the recent generalization is to be found in [202, 2010] by Milan Janjić, where the number of all generalized compositions of a natural number is a weighted r -generalized Fibonacci number introduced in [203, 2008] by Matthias Schork.

For *example 1*: F_{n+1} equals to the number of ways to cover a $2 \times n$ checkerboard with 2×1 dominoes. There exist also - apart from Listing. **14**. - at least a twenty years young poset interpretation.

For *example 2*: the n -th Fibonacci number, F_n may be interpreted as the number of ideals in a fence poset as exploited in [156, 1990] by István Beck. For

further continuation of relevant investigation of fences and crowns see: [157, 1991], [158, 1992], [159, 1995] by Jonathan David Farley, [160, 2002], [162, 2003], [161, 2004] by Emanuelle Munarini and Norma Zagaglia Salvi, [163, 2008] by Rodolfo Salvi and Norma Zagaglia Salvi, [164, 2009] by Alessandro Conflitti.

- What - if extended? The tiling interpretation of Fibonacci-like numbers is known since several years - up to the knowledge of the present author.

For *example 3*: according to the tilings' Combinatorial Theorem 5, p.36 in [132, 2003] by Arthur T. Benjamin and Jennifer J. Quinn we ascertain that for $H_n = U_n$ the number s from this note recurrence (2) is interpreted as equal to the number of colors of squares and t from this very recurrence (2) equals to the number of colors of dominos while $H_n = U_{n+1}$ counts colored tilings of length n with squares and dominos.

The method of proving various identities in tiling language with the {**mononimo**, **domino**,}-alphabet (\equiv Morse words alphabet) for number sequences was developed also by Arthur T. Benjamin and Jennifer J. Quinn and convoked partly in [132, 2003]. It is becoming now a quite popular tool; for instance see [165, 2008] by Mark Shattuck or [166, 2009] by Matt Katz and Catherine Stenson.

It is worthy to ascertain here the pleiad of investigations and enjoyment with **polyominoes** (Pólya-ominoes ?) and Pólya Festoons - so named in [169, 1991] by Philippe Flajolet all these to be placed side by side with the source idea from [140, 1969] by György Pólya (see more: [Ex. q* ; 2] and [Ex. q* ; 3] below in subsection 4.3.).

Just to make a glimpse at this pleiad we evoke some references - being totally incomplete with this undertaking. Here these are: [167, 1981] by C. Berge, C. C. Chen, Vasek Chvátal and S. C. Seow , [168, 1990] by J. H. Conway and J. C. Lagarias, [170, 1991] by Maylis Delest, [171, 1991] by Maylis Delest, [172, 1995] by Maylis Delest, J. P. Dubernard and I. Dutour, [174, 1999] by Dean Hickerson , [173, 1999] by E. Barcucci, A. Del Lungo, E. Pergola and Renzo Pinzani, [175, 2003] by A. Del Lungo, E. Duchi, A. Frosini, S. Rinaldi; see also references in [http : //mathworld.wolfram.com/PolyominoTiling.html](http://mathworld.wolfram.com/PolyominoTiling.html) and [http : //mathworld.wolfram.com/Polyomino.html](http://mathworld.wolfram.com/Polyomino.html)

Let us also note numerous papers by Svjetlan Feretić et all. just to mention one [176, 2004] by Svjetlan Feretić - directly referring to György Pólya [quote] *In 1938, Pólya stated an identity involving the perimeter and area generating function for parallelogram polyominoes. To obtain that identity, Pólya presumably considered festoons. A festoon (so named by Flajolet) is a closed path w which can be written as $w = uv$, where each step of u is either $(1, 0)$ or $(0, 1)$, and each step of v is either $(-1, 0)$ or $(0, -1)$.* [end of quote] - (see [Ex. q* ; 2] and [Ex. q* ; 3] below in subsection 4.3.).

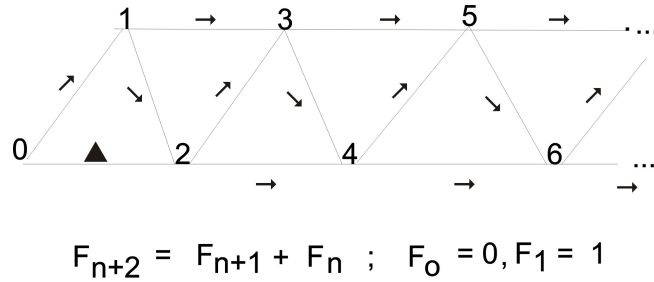
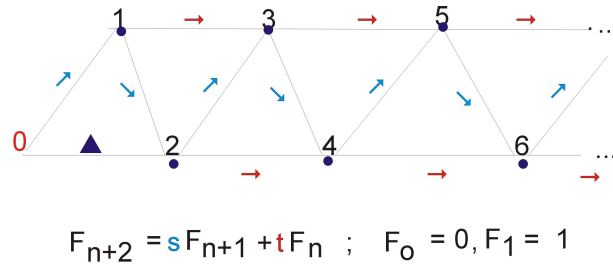


Figure 6: Here are - back to Home **n** from the Tavern **0** - possible night paths - marked with shining arrows.

For *example 4*: the n - *th* Fibonacci number F_n may be interpreted as the number of ways to get back Home **n** from the Tavern **0** along paths - *marked with arrows* as displayed by the self-explanatory Fig.6. (Attention: by paths we mean here paths - *marked with arrows* i.e at least one arrow.)

For *example 5*: the n - *th* weighted Fibonacci number i.e. $F_n \equiv U_n(s, t)$ may be interpreted as the number of - back to Home **n** from the Tavern **0** - possible weighted paths - marked with red **t** weighted horizontal arrows and with blue **s** weighted sloping arrows as displayed by the self-explanatory Fig.7. **Attention**: by paths we mean here paths - *marked with arrows* i.e at least with one arrow. In this way there is no way out, there is no path in case Home = Tavern i.e. there is no problem in reasonable case of Home \neq Tavern.



→ **weight**
↗ ↘ **weight**

Figure 7: Here are - back to Home **n** from the Tavern **0** - possible weighted paths - marked with red **t** weighted horizontal arrows and with blue **s** weighted sloping arrows.

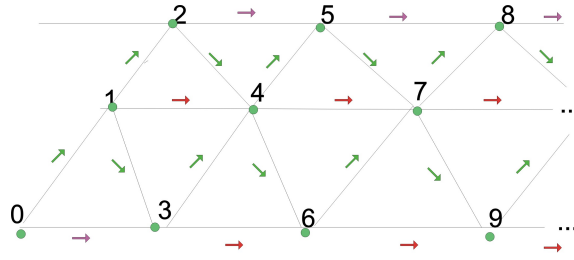


Figure 8: Here are - back to Home **n** from the Tavern **0** - possible two strip highway paths - marked with guiding horizontal and sloping arrows.

For *example 6*: the n - *th* two strip highway Fibonacci-like number from the self-explanatory Fig.8 may be interpreted as the number of - back to Home **n** from the Tavern **0** - possible paths - marked with horizontal and sloping arrows as displayed by the self-explanatory Fig.8.

According to the Fig.8 we have in this case the following recurrent relations: $F_0 = 0$, $F_1 = 1$, $F_2 = 2$ and $F_{n+3} = F_{n+2} + F_{n+1} + F_n$ for $n = 1 + k \cdot 3$, $k \geq 0$, $F_{n+3} = F_{n+1} + F_n$ for $n = 0 + k \cdot 3$, $k \geq 0$ and $F_{n+3} = F_{n+2} + F_n$ for $n = 2 + k \cdot 3$, $k \geq 0$.

Equivalently: $F_{n+3} = F_{n+2} + F_{n+1} + F_n = 1 \bmod 3$ for $n = 1 + k \cdot 3$, $k \geq 0$, $F_{n+3} = F_{n+1} + F_n = 0 \bmod 3$ for $n = 0 + k \cdot 3$, $k \geq 0$ and $F_{n+3} = F_{n+2} + F_n = 2 \bmod 3$ for $n = 2 + k \cdot 3$, $k \geq 0$.

Tiling-combinatorial Interpretation of U -binomials Question.

What is then combinatorial interpretation of U -binomial coefficients stemming from the above different languages examples? Does notation of items' representing - matter?

Are the interpretations proposed in [46, 2009] by Bruce E. Sagan and Carla D. Savage (Listing. **12.**) and/or in [136, 2010] by Johann Cigler (Listing. **11.**) the simplest answer to *Tiling-combinatorial Interpretation of U -binomials Question*?

As a **Motto** for an experienced answer we propose:

Science is a language (from [205, 1996] by Doron Zeilberger);

Mathematical notation evolves like all languages (from [206, 1992] by Donald Ervin Knuth).

Another attempt related to *example 3* above by by Arthur T. Benjamin and Sean S. Plott in [133, 2010] is presented in the subsequent subsection **4.2**.

The above list is open and far from complete.

4.2.

Nevertheless, to this end **let us discern** in part - via indicative information - a part of **Arthur T. Benjamin and coworkers'** recent contribution to the tiling interpretation domain. **Firstly**, let us track in [132, 2003] the tilings' Combinatorial Theorem 5, p.36. There for $H_n = U_n$ the number s from this note recurrence (2) is interpreted as equal to the number of colors of squares and t from this very recurrence (2) equals to the number of colors of dominos while $H_n = U_{n+1}$ counts colored tilings of length n with squares and dominos. Similarly - also in [132, 2003] by Arthur T. Benjamin and Jennifer J. Quinn let us consider the tilings' Combinatorial Theorem 6, p.36. Here for $H_n = V_n$ the number s from this note recurrence (2) should equal to the number of colors of a square and t from this very recurrence (2) equals to the number of colors of a domino while $H_n = V_n$ counts colored bracelets of length n tiled with squares and dominos. Bruce E. Sagan and Carla D. Savage in [46, 2009] refer to well known recurrences: Identity 73 on p. 38 in [132] - for (4) in [46] and Identity 94 p. 46 in [132] for (5) in [46]. Both (4) and (5) recurrences in [46, 2009] by Bruce E. Sagan and Carla D. Savage have been evoked in the illustrative Example 3. Section 3. above.

Firstly, the paper [133, 2009] by Arthur T. Benjamin and Sean S. Plott referring to [132, 2003] by Arthur T. Benjamin and Jennifer J. Quinn should be notified as it proposes a new formula for U -binomials derived via tiling argumentation. Being occasionally nominated by Arthur T. Benjamin and Sean S. Plott in errata [133, 2010] to this paper the present author feels entitled to remark also on this errata.

According to errata [133, 2010] by Arthur T. Benjamin and Sean S. Plott [quote] " The formula for $\binom{n}{k}_F$ should be multiplied by a factor of F_{n-x_k} , which accounts for the one remaining tiling that follows the f_0 tiling. Likewise, the formula for $\binom{n}{k}_F$ should be multiplied by U_{n-x_k} . " Our remark is that this errata is unsuccessful. If we follow this errata then $(x_{k-1} < x_k)$ we would have:

$$(48) \quad \binom{n}{k}_{errata} = \sum_{1 \leq x_1 < x_2 < \dots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i - (k-i) + 1} F_{n-x_k},$$

where $F_0 = 0$ and $x_0 = 0$. But the formula (48) implies for example

$$15 = \binom{5}{3}_F \neq \binom{5}{3}_{errata} = 11.$$

The task of finding the correct formula - due to the present author became two months ago an errand - exercise for Maciej Dziemiańczuk, a doctoral student from Gdańsk University in Poland. The result - to be quoted below as MD formula (50) - is his discovery, first announced in the form of a feedback private communication to the present author: (M. Dziemiańczuk on Mon, Oct 18, 2010 at 6:26 PM) however still not announced in public.

The source of an error in errata is that $\binom{n}{k}_F$ should be multiplied **not by** the factor of F_{n-x_k} *but by* the factor $F_{n-x_k+1} \equiv f_{n-x_k}$. Then we have

$$\binom{n}{k}_{now} = \sum_{1 \leq x_1 < x_2 < \dots < x_{k-1} \leq n-1} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i - (k-i) + 1} F_{n-x_k + 1},$$

Due to $x_{k-1} < x_k$ the above formula is equivalent to

$$(49) \quad \binom{n}{k}_{now} = \sum_{1 \leq x_1 < x_2 < \dots < x_{k-1} < x_k \leq n} \prod_{i=1}^{k-1} F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i - (k-i) + 1} F_{n-x_k},$$

and this in turn is evidently equivalent to the MD-formula (50) below i.e. (49) is equivalent to the corrected by Maciej Dziemiańczuk Benjamin and Plott formula from The Fibonacci Quarterly 46/47.1 (2008/2009), 7-9.

Finally here now MD-formula follows:

$$(50) \quad \binom{n}{k}_F = \sum_{1 \leq x_1 < x_2 < \dots < x_k \leq n} \prod_{i=1}^k F_{k-i}^{x_i - x_{i-1} - 1} F_{n-x_i - (k-i) + 1},$$

where $F_0 = 0$ and $x_0 = 0$.

Collaterally Maciej Dziemiańczuk supplies correspondingly correct formula for Lucas U - binomial coefficients $\binom{n}{k}_U$:

$$(51) \quad \binom{n}{k}_U = \sum_{\substack{1 \leq x_1 < x_2 < \dots < x_{k-1} \leq n \\ x_k = x_{k-1} + 1}} s^{x_k - k} \left(\prod_{i=1}^{k-1} U_{k-i}^{x_i - x_{i-1} - 1} U_{n-x_i - (k-i) + 1} \right) U_{n-x_k + 1}$$

$$(52) \quad = \sum_{1 \leq x_1 < x_2 < \dots < x_k \leq n} s^{x_k - k} \prod_{i=1}^k U_{k-i}^{x_i - x_{i-1} - 1} U_{n-x_i - (k-i) + 1},$$

where $U_0^t = 0^t = \delta_{t,0}$.

4.3. p, q -binomials versus q^* -binomials combinatorial interpretation, where $q^* = \frac{p}{q}$ if $q \neq 0$.

In the first instance let us once for all switch off the uninspired $p \cdot q = 0$ case. Then obligatorily either $q \neq 0$ or $q = 0$. Let then $q^* = \frac{p}{q}$. In this nontrivial case

$$(53) \quad \binom{n}{k}_{p,q} = q^{k(n-k)} \cdot \binom{n}{k}_{q^*}.$$

Referring to the factor $q^{k(n-k)}$ as a kind of **weight**, one may transfer combinatorial interpretation statements on q^* binomials $\binom{n}{k}_{q^*}$ onto combinatorial interpretation statements on p, q binomials $\binom{n}{k}_{p,q}$ through the agency of (49). Thence, apart from specific combinatorial interpretations uncovered for the class or subclasses of p, q -binomials there might be admitted and respected the "q*-overall" combinatorial interpretations transferred from $1, q^*$ -binomials i.e. from q^* -binomials onto p, q -binomials.

By no means pretending to be the complete list here comes the skeletonized list of **[Ex. q* ; k]** examples, $k \geq 1$.

[Ex. q* ; 1] The q^* -binomial coefficient $\binom{m+n}{m,n}_{q^*}$ may be interpreted as a polynomial in q^* whose q^{*k} -th coefficient counts the number of distinct partitions of k elements which fit inside an $m \times n$ rectangle - see [138, 1976] by George Eyre Andrews.

On lattice paths' techniques - Historical Remark. It seems to be desirable now to quote here information from [142, 2010] by Katherine Humphreys based on [143, 1878] by William Allen Whitworth:

Quotation 2 *We find lattice path techniques as early as 1878 in Whitworth to help picture a combinatorial problem, but it is not until the early 1960's that we find lattice path enumeration presented as a mathematical topic on its own. The number of papers pertaining to lattice path enumeration has more than doubled each decade since 1960.*

[Ex. q* ; 2]

The [Ex. q* ; 2] may be now compiled with [Ex. q* ; 1] above. For that to do recall that zigzag path is the shortest path that starts at $A = (0, 0)$ and ends in $B = (k, n - k)$ of the $k \times (n - k)$ rectangle; see: [139, 1962] by György Pólya [pp. 68-75], [139, 1969] by György Pólya and [141] by György Pólya and G. L. Alexanderson.

Let then $A_{n,k,\alpha}$ = the number of those $(0, 0) \rightarrow (k, n - k)$ zigzag paths the area under which is α .

In [140, 1969] György Pólya using recursion for q^* -binomial coefficients proved that

$$\binom{n}{k}_{q^*} = \sum_{\alpha=0}^{k(n-k)} A_{n,k,\alpha} \cdot q^{*\alpha}.$$

from where György Pólya infers the following Lemma ([140, 1969], p.105) which is named Theorem (p. 104) in more detailed paper [141, 1971] by György Pólya and G. L. Alexanderson.

Quotation 3 *The number of those zigzag paths the area under which is α equals $A_{n,k,\alpha}$.*

[Ex. q^* ; 3]

The [Ex. q^* ; 3] may be now compared with [Ex. q^* ; 1]. The combinatorial interpretation of $\binom{r+s}{r,s}_{q^*}$ from [Ex. q^* ; 1] had been derived (pp. 106-107) in [141, 1971] by György Pólya and G. L. Alexanderson, from where - with advocacy from [144, 1971] by Donald Ervin Knuth - we quote the result.

(1971): $\binom{r+s}{r,s}_{q^*} = \text{ordinary generating function in } \alpha \text{ powers of } q^* \text{ for partitions of } \alpha \text{ into exactly } r \text{ non-negative integers none of which exceeds } s,$

as derived in [141, 1971] by György Pólya and G. L. Alexanderson - see formula (6.9) in [141].

(1882): $\binom{n}{k}_{q^*} = \text{ordinary generating function in } \alpha \text{ powers of } q \text{ for partitions of } \alpha \text{ into at most } k \text{ parts not exceeding } (n - k),$

as recalled in [144, 1971] by Donald Ervin Knuth and proved combinatorially in [145, 1882] by James Joseph Sylvester.

Let nonce : $r + s = n, r = k$ then **(1971)** \equiv **(1882)** are equal due to

$$(54) \quad \binom{n}{k}_{q^*} = \sum_{\alpha=0}^{k(n-k)} A_{n,k,\alpha} \cdot q^{*\alpha} = \sum_{\alpha=0}^{r \cdot s} A_{r+s,r,\alpha} \cdot q^{*\alpha} = \binom{r+s}{r,s}_{q^*}.$$

where for commodity of comparison formulas in two notations from two papers - we have been using contractually for a while: $r + s = n, r = k$ identifications.

[Ex. q^* ; 4]

The following was proved in [146, 1961] by Maurice George Kendall and Alan Stuart (see p.479 and p.964) and n [141, 1971] by György Pólya and Gerald L. Alexanderson (p.106).

The area under the zigzag path = The number of inversions in the very zigzag path coding sequence.

The possible extension of the above combinatorial interpretation onto three dimensional zigzag paths via "three-nomials" was briefly mentioned in [141, 1971] - see p.108.

[Ex. q^* ; 5]

The well known (in consequence - *finite geometries'*) interpretation of $\binom{n}{k}_{q^*}$ coefficient due to Jay Goldman and Gian-Carlo Rota from [147, 1970] is now worthy of being recalled; see also [144, 1971] by Donald Ervin Knuth.

Let V_n be an n -dimensional vector space over a finite field of q^* elements. Then

$\binom{n}{k}_{q^*} = \text{the number of } k\text{-dimensional subspaces of } V_n .$

[Ex. q* ; 6]

This example = the short substantial note [144, 1971] by Donald Ervin Knuth. Compile this example with the example [Ex. q* ; 5] above.

The essence of a coding of combinatorial interpretations via bijection between lattices is the construction of this *coding bijection* in [144]. Namely, let $GF(q^*)$ be the Galois field of order q^* and let $V_n \equiv V = GF(q^*)^n$ be the n -dimensional vector space over $GF(q^*)$. Let $[n] = \{1, 2, \dots, n\}$. Let $\ell(V)$ be the lattice of all subspaces of $V = GF(q^*)^n$ while $\ell([n]) \equiv 2^{[n]}$ denotes the lattice of all subsets of $[n]$.

In [144] Donald Ervin Knuth constructs this *natural order and rank preserving* map Φ from the lattice $\ell(V)$ of subspaces onto the lattice $\ell([n]) \equiv 2^{[n]}$ of subsets of $[n]$.

$$\ell(V) \xrightarrow{\Phi} \ell([n]).$$

We bethink with some reason whether this Φ bijection coding might be an answer to the subset-subspace problem from subset-subspace problem from [127, 1998] by John Konvalina ?

Quotation 4 ...the subset-subspace problem (see 6 , 9 , and 3) . The traditional approach to the subset-subspace problem has been to draw the following analogy: the binomial $\binom{n}{k}_F$ coefficient counts k -subsets of an n -set, while the analogous Gaussian $\binom{n}{k}_q$ coefficient counts the number of k -dimensional subspaces of an n -dimensional finite vector space over the field of q elements. The implication from this analogy is that the Gaussian coefficients and related identities tend to the analogous identities for the ordinary binomial coefficients as q approaches 1. The proofs are often algebraic or mimic subset proofs. But what is the combinatorial reason for the striking parallels between the Gaussian coefficients and the binomial coefficients?

According to Joshef P. S. Kung [148, 1995] the Knuth's note is not the explanation:

Quotation 5 ... observation of Knuth yields an order preserving map from $L(V_n(q))$ to Boolean algebra of subsets, but it does not yield a solution to the still unresolved problem of finding a combinatorial interpretation of taking the limit $q \rightarrow 1$.

Well, perhaps this limit being performed by q -deformed Quantum Mechanics physicists might be of some help? There the so called q -quantum plain of q -commuting variables $x \cdot y - q \cdot y \cdot x = 0$ becomes a plane $\mathbb{F} \times \mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \dots$ p -adic fields included) of two commuting variables in the limit $q \rightarrow 1$. For see [149, 1953] by Marcel-Paul Schützenberger. For quantum plains - see also [150,

1995] by Christian Kassel. It may deserve notifying that q - extension of the "classical plane" of commuting variables ($q = 1$) seems in a sense ultimate as discussed in [151, 2001] by A.K. Kwaśniewski

[Ex. q^* ; 7]

Let us continue the above by further quotation from [127, 1998] on generalized binomial coefficients and the subset-subspace problem.

Quotation 6 *We will show that interpreting the Gaussian coefficients as generalized binomial coefficients of the second kind combinations with repetition reveals the combinatorial connections between not only the binomial coefficients and the Gaussian coefficients, but the Stirling numbers as well. Thus, the ordinary Gaussian coefficient tends to be an algebraic generalization of the binomial coefficient of the first kind, and a combinatorial generalization of the binomial coefficient of the second kind.*

Now in order to get more oriented go back to the beginning of subsection **4.1.** and consult : Listing. **1.**, Listing. **2.**, Listing. **3.** which are earlier works and end up with [128, 2000] by John Konvalina on an unified simultaneous interpretation of binomial coefficients of both kinds, Stirling numbers of both kinds and Gaussian binomial coefficients of both kinds. **Compare it** then thereafter with Listing. **8.**

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